ON THE NON-SELF-ADJOINT OPERATOR \(-y”+p(x)y\) ON THE AXIS \((-\infty,\infty)\)

MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 517.91

MATHEMATICS

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ON THE NON-SELF-ADJOINT OPERATOR $-y'' + p(x)y$ ON THE AXIS $(-\infty, \infty)$

(Presented by Academician A. A. Dorodnitsyn, 21 XI 1969)

We consider the differential equation

$$-y'' + p(x)y = \lambda y, \tag{1}$$

where $p(x) = q(x) + ir(x)$ is a complex-valued function of the real argument x, and λ is a complex parameter. Associated with equation (1) is the non-self-adjoint differential operator acting in $L^2(-\infty, \infty)$,

$$Ly = -y'' + p(x)y,$$

whose domain $D(L)$ consists of functions $y(x) \in L^2(-\infty, \infty)$ that are absolutely continuous on every finite interval together with their derivative and such that $-y'' + p(x)y \in L^2(-\infty, \infty)$.

In the present paper, with the aid of an auxiliary system of nonlinear first-order differential equations, we investigate the question of the existence of nontrivial solutions of equation (1) belonging to $L^2(-\infty, \infty)$, and give conditions imposed on $p(x)$ under which the operator L has a completely continuous resolvent and, consequently, a discrete spectrum. The results presented in the paper are adjacent to certain results of M. A. Naimark ([1]) and V. B. Lidskii ([2]).

Consider the equation

$$-y'' + p(x)y = 0 \tag{2}$$

and suppose that for $x \in [a, b]$ the function $p(x)$ can be represented in the form

$$p(x) = \rho(x)e^{i\varphi_0(x)},$$

where $\rho(x)$ and $\varphi_0(x)$ are continuous functions, and

$$\rho(x) \geq \rho_0 > 0. \tag{3}$$

Make in (2) the substitution

$$y' = y \operatorname{ctg} \theta(x) e^{i\varphi(x)}.$$

This substitution is an analogue of Prüfer' s substitution ([3]), often used in the study of the self-adjoint case. For the functions $\theta(x)$ and $\varphi(x)$ we obtain the system of nonlinear equations

$$\theta' = \cos^2\theta \cos\varphi - \sin^2\theta\, \rho(x) \cos(\varphi - \varphi_0(x)),$$

$$\varphi' = -\operatorname{ctg}\theta \sin\varphi - \operatorname{tg}\theta\, \rho(x) \sin(\varphi - \varphi_0(x)). \tag{4}$$

Theorem 1. *To every nontrivial solution $y(x)$ of equation (2) there corresponds a pair of real continuously differentiable functions $\theta(x)$ and $\varphi(x)$ on the interval $[a, b]$ such that*

$$y'(x)/y(x) = \operatorname{ctg}\theta(x) e^{i\varphi(x)}.$$

The functions $\theta(x)$ and $\varphi(x)$ satisfy system (4) and, moreover, possess the following properties:

1) $\theta(x)$ can cross the lines $\theta = k\pi$ from below upward only at those points where $\varphi(x) = 2m\pi$, and from above downward only at those points where $\varphi(x) = (2m+1)\pi$.
2) $\theta(x)$ can cross the lines $\theta = \pi/2 + k\pi$ from below upward only at those points where $\varphi(x) = \varphi_0(x) + (2m+1)\pi$, and from above downward only at those points where $\varphi(x) = \varphi_0(x) + 2m\pi$.

The functions $\theta(x)$ and $\varphi(x)$ are determined uniquely up to the transformations

$$\theta = \theta + n\pi; \qquad \varphi = \varphi + 2n\pi; \qquad \theta = -\theta,\ \varphi = \varphi + \pi.$$

Suppose that the function $p(x) = \rho(x)e^{i\varphi_0(x)}$ is such that, for $x \in [a, b]$, its values lie in an open angle whose closure does not contain the negative real semiaxis, i.e., for $x \in [a, b]$,

$$\gamma < \arg\varphi_0(x) < \delta, \tag{5}$$

where $-\pi < \gamma < 0,\ 0 < \delta < \pi$.

Lemma 1. Let the function $p(x)$ satisfy conditions (3) and (5) on the segment $[a,b]$, with $\delta - \gamma \le \pi$. Then, if the functions $\theta(x)$ and $\varphi(x)$ are solutions of system (4) such that

$$0 < \theta(a) < \pi/2, \qquad \gamma < \varphi(a) < \delta,$$

then for all $x \in [a,b]$ the relations

$$0 < \theta(x) < \pi/2, \qquad \gamma < \varphi(x) < \delta$$

hold.

If $p(x)$ is continuously differentiable for sufficiently large values of $|x|$, satisfies conditions (3) and (5) on the interval $(-\infty,\infty)$, and

$$\lim_{|x|\to\infty} p'(x)/p^{3/2}(x) = 0, \tag{6}$$

then equation (2) has solutions $\chi(x)$ and $\psi(x)$, unique up to constant factors, belonging respectively to $L^2(-\infty,0)$ and $L^2(0,\infty)$.

Moreover,

$$\chi'(x)/\chi(x) = \sqrt{p(x)}(1+o(1)), \qquad x \to -\infty, \tag{7}$$

$$\psi'(x)/\psi(x) = -\sqrt{p(x)}(1+o(1)), \qquad x \to +\infty. \tag{8}$$

The value of $\sqrt{p(x)}$ is chosen in the right half-plane.

Using Lemma 1 and the asymptotic formulas (7) and (8), one can prove the following lemma.

Lemma 2. Let the function $p(x) = \rho(x)e^{i\varphi_0(x)}$ be continuously differentiable on the interval $(-\infty,\infty)$, satisfy conditions (3), (5), and (6) on this interval, and let, moreover, for $x \in (-\infty,\infty)$,

$$|\varphi_0'(x)|/\sqrt{\rho(x)} \le d,$$

where $d = \min(|\sin\gamma|, \sin\delta, \cos\gamma/2, \cos\delta/2)$. Then the solutions $\chi(x)$ and $\psi(x)$ are linearly independent, and equation (2) has no nontrivial solutions belonging to $L^2(-\infty,\infty)$.

Put

$$V_c^{\gamma,\delta} = \{z : \gamma < \arg(z-c) < \delta\}.$$

From Lemma 2 it follows:

Theorem 2. Let the function $p(x)$ in the equation

$$-y'' + p(x)y = \lambda y \tag{9}$$

satisfies condition (6), and suppose there exists a real c such that for $x \in (-\infty, \infty)$

$$p(x) \in V_c^{\gamma,\delta},$$

where $-\pi < \gamma < 0,\ 0 < \delta < \pi$. Then there exists a real number λ_0 such that, for real $\lambda \leq \lambda_0$, equation (9) has no nontrivial solutions belonging to $L^2(-\infty, \infty)$.

When the conditions of Theorem 2 are fulfilled, on the real half-axis $\lambda \leq \lambda_0$ there are no points of the discrete spectrum of the operator L. The following theorem gives a sufficient condition for discreteness of the spectrum of the operator L.

Theorem 3. Suppose the function $p(x) = q(x) + ir(x)$ satisfies condition (6), and suppose there exist real constants k^- and k^+ such that

$$\lim_{x \to -\infty} (q(x) + k^- r(x)) = +\infty,$$

$$\lim_{x \to +\infty} (q(x) + k^+ r(x)) = +\infty.$$

Then the operator

$$Ly = -y'' + p(x)y$$

has a completely continuous resolvent and, consequently, a purely discrete spectrum.

We note that under the assumptions of Theorem 3 the set of values of the quadratic functional (Ly, y) may, in contrast to the cases considered in (2), fill the entire plane.

The author expresses his gratitude to A. A. Abramov for useful advice and to V. B. Lidskii for discussion of the results.

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Received
13 XI 1969

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Note: Figure translations are in progress. See original paper for figures.