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Abstract

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NORMAL CLASSES OF SMOOTH IMMERSIONS AND EMBEDDINGS

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The present note is devoted to two problems connected with Euler normal classes. The first problem consists in enumerating the normal classes of immersions of a manifold in a manifold that are homotopic to a given immersion; the second, in enumerating the normal classes of embeddings of a closed nonorientable manifold in Euclidean space (let us recall that in the orientable case these classes are zero).

All manifolds and their mappings considered are assumed to be C^∞-smooth.

1. Normal classes of immersions. Lashof and Smale in (^{2}) computed the normal classes of immersions of a closed orientable n-dimensional manifold in R^{2n}. A further investigation was carried out in a recent work of Becker (^{1}), who described, in the metastable case, the group of differences of normal classes of immersions of a closed orientable manifold in Euclidean space modulo the class of finite abelian groups.

Let M be a closed n-dimensional manifold, N an orientable $(n+k)$-dimensional manifold without boundary, and $f : M \to N$ an immersion. In this paragraph we shall give, for the metastable case, a description of the normal classes of immersions $M \to N$ homotopic to the given immersion f.

We shall denote by Z_M the local system of integral coefficients compatible with the tangent bundle $\tau(M)$ of the manifold M, by X_f the Euler normal class of the immersion f, by w_f the stable Stiefel-Whitney normal class of dimension k (with coefficients in Z_M in the case of odd k), and by $\mathfrak{N}_f$ the set of elements of the group $H^k(M; Z_M)$ realizable as normal classes of immersions homotopic to f.

The only nontrivial case in our problem is that of even codimension k. For odd k the question is settled by the simple Theorem 1.

Theorem 1. Let M be a closed n-dimensional manifold; N an orientable $(n+k)$-dimensional manifold without boundary, and $f : M \to N$ an immersion. If $k > 1$ and is odd, and $g : M \to N$ is an immersion homotopic to f, then

$$X_g = w_f.$$

A special case of this theorem is contained in (^{1}).

For the manifold M, denote by $T(M)$ the Thom space of the tangent bundle, by $T_f(M)$ the Thom space of the bundle $\tau_f(M)$ induced from the tangent bundle by the mapping f, and by U_f the Thom class of the bundle $\tau(M)$.

Let $i_1 : M \to T(M)$, $i_2 : M \to T_f(M)$ be the zero sections, and let $\mathrm{Hom}(T(M); T_f(M))$ be the set of such fiber-preserving mappings $\alpha : T(M) \to T_f(M)$ that $\alpha i_1 = i_2$. Define a mapping

$$\eta : \mathrm{Hom}(T(M); T_f(M)) \to H^k(M, Z_M)$$

by putting $\eta(\varphi) = t_M^{-1}\varphi^*(U_f)$, where

$$t_M : H^k(M, Z_M) \to H^{n+k}(T(M); Z)$$

is the Thom isomorphism. Denote by A_f the set $\mathrm{im}(\eta)$. It is easy to verify

Proposition 1. Let

$$D : H_{n-k}(M; Z) \to H^k(M; Z_M)$$

be the Poincaré duality operator. If $2k \geq n+1$, then the class y is contained in $D^{-1}(A_f)$ if and only if it is realized by a submanifold L, whose normal bundle in M is equivalent to the restriction to L of the normal bundle ν_f of the immersion f.

From Proposition 1 and the preceding general considerations it follows that, for $2k \geq n+1$, the set A_f is a subgroup of the group $H^k(M; \mathbb{Z}_M)$. It is also clear that A_f does not depend on the choice of the immersion f in the given homotopy class.

The main result of this paragraph is

Theorem 2. Let M be an orientable, closed, n-dimensional manifold, let N be an orientable $(n+k)$-dimensional manifold without boundary, and let $f : M \to N$ be an immersion. If $2k \geq n+1$, then $\mathfrak{N}_f = X_f + 2A_f$.

The proof of the inclusion $X_f + 2A_f \subset \mathfrak{N}_f$ is based on Lemma 1. In order to formulate it, we introduce the following notation. Let L be an orientable, closed manifold and let ν be a smooth vector bundle of even dimension k over it. Fix in the bundle $\nu \oplus \varepsilon^1$, where ε^1 is the trivial line bundle, a Riemannian metric, and denote by B_ν the manifold of unit vectors of this bundle and by U_ν the corresponding Thom class. By $E_{\nu\oplus\nu}$ we denote the total space of the bundle $\nu \oplus \nu$.

Lemma 1. There exists an immersion $s(\nu) : B_\nu \to E_{\nu\oplus\nu}$ for which $X_{s(\nu)} = 2U_\nu$.

With the help of the immersions indicated in Lemma 1, under the dimensional assumptions of Theorem 2 one can perform modifications of the initial immersion f along special $(n-k)$-dimensional submanifolds, thereby changing the class X_f in this way.

The proof of the reverse inclusion is based on the Lashoof-Smale formula (^{3}), which relates the normal Euler class to the homology class determined by the self-intersection manifold, and uses the reduction of this manifold to a simple form.

Let us note that the inclusion $X_f + 2A_f \subset \mathfrak{N}_f$ also holds in the case when the manifold M is nonorientable. The author does not know whether, for nonorientable M, the reverse inclusion holds.*

We formulate some consequences of Theorem 2. Below, for a space Y, $S^i(Y)$ denotes the set of spherical elements of the group $H^i(Y; \mathbb{Z})$.

Theorem 3. Let M be an orientable closed n-dimensional manifold; let N be a manifold of dimension $n+k$, and let $f : M \to N$ be an immersion. If k is even and $2k \geq n+1$, and the manifold N is $(n-k)$-parallelizable, then

$$\mathfrak{N}_f = X_f + 2t_M^{-1}(S^{n+k}(T(M))).$$

A special case of this theorem is

Theorem 4. Let M be a closed orientable n-dimensional manifold and let $f : M \to R^{n+k}$ be an immersion. If k is even and $2k \geq n+1$, then

$$\mathfrak{N}_f = X_f + 2t_M^{-1}(S^{n+k}(T(M))).$$

For the case when the manifold M is parallelizable, Theorem 4 was proved by Becker (^{1}).

2. Normal classes of embeddings of nonorientable manifolds in Euclidean space.

The problem of enumerating normal classes of embeddings of nonorientable manifolds in Euclidean space was considered in the papers ($^{4-8}$).

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* **Note added in proof.** At present, the result of Theorem 2 has been obtained by the author also in the case when the manifold M is nonorientable. Moreover, the following general result has been obtained.

Let K be a finite n-dimensional cell complex, let ν be a k-dimensional vector bundle over it, and let ν^{-1} be an $(n+1)$-dimensional vector bundle such that

$$\nu \oplus \nu^{-1} = \varepsilon^{n+k+1},$$

and let $T(\nu^{-1})$ be the total space of the bundle of n-dimensional spheres associated with the bundle ν^{-1}. If the number k is even and $2k \geq n+1$, then the set of Euler classes of k-dimensional bundles stably equivalent to ν coincides with the set

$$X_\nu + 2t^{-1}(S^{n+k}(T(\nu^{-1}))).$$

The information contained there pertains only to embeddings in a space of twice the dimension.

In this paragraph we formulate two new results. One concerns the metastable case and describes the set of normal classes of embeddings up to groups of finite order; the other gives an exact enumeration of the normal classes of embeddings of a closed manifold of dimension $n \geqslant 5$ in R^{2n-1}.

Let M be a closed, nonorientable manifold and $f : M \to R^{n+k}$ an embedding. By $\widetilde{M}$ we denote the deleted square of the manifold M, and by $D(M)$ the total space of the bundle of closed n-dimensional balls associated with the tangent bundle $\tau(M)$.

The inclusion of pairs

$$i: \ (D(M), \partial D(M)) \to (M \otimes M, \widetilde{M})$$

gives a commutative diagram

$$\begin{tikzcd}
\to H^{n+k-1}(\widetilde{M}; Z) \arrow[r, "\delta"] \arrow[d, "i_3^*"] & H^{n+k}(M \times M, \widetilde{M}; Z) \arrow[r, "\tilde{j}^*"] \arrow[d, "i_1^*"] & H^{n+k}(M \times M; Z) \to \arrow[d, "i_2^*"] \\
\to H^{n+k-1}(\partial D(M); Z) \arrow[r, "\delta"] & H^{n+k}(D(M), \partial D(M); Z) \arrow[r, "j^*"] & H^{n+k}(D(M); Z) \to .
\end{tikzcd}$$

We note that here

$$i_1^* : H^{n+k}(M \times M, \widetilde{M}; Z_M) \to H^{n+k}(D(M), \partial D(M); Z)$$

is an isomorphism. Denote by P_k the kernel of the homomorphism

$$\tilde{j}^*(i_1^*)^{-1} t_M : H^k(M; Z_M) \to H^{n+k}(M \times M; Z).$$

Theorem 5. Let $2k \geqslant n + 3$ and let the number k be even. Then the set of normal classes of embeddings of the manifold M in R^{n+k} is a coset of the group P_k modulo a subgroup of finite index.

The method of proof is purely algebraic and is analogous to the method used by Becker in (^{1}). An exact description of the set of normal classes of embeddings by this method is not obtained in any substantial case. We note that the case of odd codimension is exhausted by Theorem 1.

We formulate two consequences of Theorem 5. Let $\Gamma^k(M)$ be the subgroups of $H^k(M; Z_M)$ Poincaré dual to the image of the Hurewicz homomorphism.

Theorem 6. For any $y \in \Gamma^k(M)$ there exist an integer a and an embedding $g : M \to R^{n+k}$ such that

$$a \cdot y = X_f - X_g.$$

Theorem 7. Suppose that, in addition to the hypotheses of Theorem 5, the manifold M is such that for every $m \leqslant (n-k)/2$ and $m \geqslant 1$ the group $H_m(M; Z)$

is finite. Then for any $y \in H^k(M; Z_M)$ there exist an integer a and an embedding $g : M \to R^{n+k}$ such that

$$a \cdot y = X_f - X_g.$$

The following result is obtained by means of geometric constructions.

Theorem 8. Let the number n be odd and $n \geqslant 5$. Then the set of normal classes of embeddings of the manifold M in R^{2n-1} coincides with the subgroup

$$2H^{n-1}(M; Z_M)$$

of the group $H^{n-1}(M; Z_M)$.

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Note: Figure translations are in progress. See original paper for figures.

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