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Abstract

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The application of the method of point transformations to the study of the dynamics of nonlinear (even piecewise-linear) distributed systems encounters considerable difficulties ([1]). In a number of works this method has been used to study the dynamics of relay systems containing a linear distributed element ([2,3]). Below we consider the possibilities of this method in analyzing the dynamics of nonlinear impulse ([4]) systems with a linear distributed element.

1. Consider, analogously to ([5]), the dynamics of a single-loop impulse control system, in whose scheme, in addition to the common feedback, there are connected in series an ideal ([4]) impulse element (with period τ), an integrating element, a static nonlinear element with characteristic $\varphi(\mu) \in C^1(R^1)$, and, for simplicity, a one-dimensional distributed element (with respect to the coordinate $a \leq x \leq b$), described by a linear homogeneous equation of parabolic type ([6]) with coefficients independent of time t and a homogeneous boundary condition at $x = b$, connected to the system in such a way that its input is the value $\varphi(\mu)$, specifying the boundary condition at $x = a$, and its output is the value of the distribution in it $u(t, x)$ at $x = l \in (a, b)$. We shall also assume that the Green' s function of the homogeneous boundary-value problem for the distributed element ([6])

$$K^1(x, y; t) \in L_2([a, b] \times [a, b]), \qquad t > 0, \tag{1}$$

i.e., satisfies the Fredholm conditions ([7,8]) for $t > 0$.

The instantaneous state of the system under consideration is determined, evidently, by the value of the coordinate of the integrating element $\mu(t')$ and by the instantaneous distribution $u(t', x)$ in the distributed element. Therefore, as the phase space of the system one may take the topological product of the μ-axis, the functional space of functions $u(x) \in C^2$ satisfying the boundary-value problem, and the half-axis $t \geq 0$. The system proves to be closed only at the instants of time $t = j\tau$ $(j = 1, 2, \ldots)$; therefore [5] it is appropriate to reduce the study of its dynamics to the study of point transformations of sections of the indicated space at $t = j\tau$ into one another.

Introducing the notation $u(j\tau - 0, x) \equiv u^j(x)$, $\mu(j\tau - 0) \equiv \mu^j$, $\varphi(\mu^j) \equiv \varphi_j$, $\varphi'_\mu(\mu^j) \equiv \varphi'_j$, and taking into account that μ and $\varphi(\mu)$ are constant on the time intervals $\theta_j \equiv (j-1)\tau \le t \le j\tau$, $j = 1, 2, \ldots$, we represent [6] the motion of the system under consideration for any of these intervals in the form

$$u(t, x) = \varphi_j f^1(t - \theta_j, x) + \int_a^b K^1(x, y; t - \theta_j) u^{j-1}(y)\, dy,$$

$$\mu(t) = \mu^0 + \sum_{j=0}^{\infty} u^j(l)\, 1(t - j\tau), \qquad (2)$$

where, by virtue of (1), $f^1(t, x) \in L_2[a, b]$ for any $t > 0$. Setting $t = j\tau$ and, for brevity, omitting the constant τ in the arguments, we obtain from (2) the indicated

above the one-fold point transformation for any j

$$u^j(x) = \varphi_j f^1(x) + \int_a^b K^1(x, y) u^{j-1}(y)\, dy, \qquad \mu^j = \mu^{j-1} + u^{j-1}(l). \qquad (3)$$

Using the iterated [7–9] kernels $K^i(x, y)$ and the functions

$$f^i(x) = \int_a^b K^{i-1}(x, y) f^1(y)\, dy,$$

the m-fold point transformation can be written in the form

$$u^j(x) = \sum_{i=1}^{j} \varphi_i f^{j-i+1}(x) + \int_a^b K^j(x, y) u^0(y)\, dy,$$

$$\mu^j = \mu_0 \sum_{i=0}^{j-1} u^i(l), \qquad j = 1, 2, \ldots, m. \qquad (4)$$

2. Periodic motions of the system with period $T = m\tau$ correspond to invariant points

$$u^0(x) = u^m(x) = \overset{*}{u}{}^m(x), \quad \mu^0 = \mu^m = \overset{*}{\mu}{}^m; \quad \overset{*}{u}{}^j(x), \overset{*}{\mu}{}^j, \quad j = 1, 2, \ldots, m-1 \qquad (5)$$

(which are not points of smaller multiplicity) of transformation (4), in which, under condition (5), the last line (for $j = m$) contains a Fredholm integral equation of the second kind. If unity, for a given τ, does not belong to the spectrum [7–10] of the kernel $K^m(x, y)$, then, solving this equation by means of the resolvent $\Gamma_m(x, y; 1)$ of this kernel, it is easy to obtain for the sought invariant point the system of equations ($f^j(x) = 0$ for $i = 0, -1, ...$)

$$\overset{*}{u}{}^j(x) = \sum_{i=1}^{m} \varphi_i \left(f^{j-i+1}(x) + f^{m+j-i+1}(x) + \int_a^b \int_a^b K^j(x,y)\Gamma_m(y,z;1) f^{m-i+1}(z)\,dz\,dy \right),$$

$$\overset{*}{\mu}{}^j = \overset{*}{\mu}{}^m + \sum_{i=0}^{j-1} \overset{*}{u}{}^i(l), \tag{6}$$

where $j = 1, 2, \dots, m-1$, and for $j = m$, in addition to what has been used,

$$\sum_{i=1}^{m} \overset{*}{u}{}^i(l) = 0.$$

3. Adding the expressions (3), written for $j = 1, 2, \dots, m$ for the invariant point (5) found with the aid of system (6), introducing the notation

$$\overset{*}{u}(x) \equiv \sum_{j=1}^{m} \overset{*}{u}{}^j(x)$$

and then putting, in particular, $x = l$, we obtain that for periodic motions of the system under consideration (apart from $\overset{*}{u}(l) = 0$) there holds, reflecting, as in [5], the symmetrizing role of the integrating element, the relation

$$\sum_{j=1}^{m} \varphi_j = 0,$$

if, for the given τ, the value l does not satisfy the equation

$$f^1(l) + \int_a^b \Gamma_1(l, y; 1) f^1(y)\,dy = 0. \tag{7}$$

Indeed, the function $\overset{*}{u}(x)$ satisfies the Fredholm integral equation of the second kind with free term

$$f^1(x)\sum_{j=1}^{m}\varphi_j$$

and kernel $K^1(x,y)$, whose resolvent enters into condition (7).

4. To investigate the stability in the small of the invariant point obtained with the aid of (6), one should linearize [2,3,11] at it the transformation

(4). However, it is simpler to obtain the system in variations ([5]) from the transformations (3).

Indeed, for convenience let us adjoin to expressions (3) also the equations for $u'(l)$ and linearize this system at the invariant point under consideration. Then, in the notation introduced below, we obtain

$$v^m = \prod_{j=m}^{1} A_j(y_j, y_{j-1})v^0, \qquad v^j \equiv \begin{pmatrix} \Delta\mu^j \\ \Delta u^j(l) \\ \delta u^j(x) \end{pmatrix},$$

$$A_j \equiv \begin{pmatrix} 1 & 1 & 0 \\ \varphi_j' f^1(l) & \varphi_j' f^1(l) & \int_a^b K^1(l, y_{j-1}) \dots dy_{j-1} \\ \varphi_j' f^1(y_j) & \varphi_j' f^1(y_j) & \int_a^b K^1(y_j, y_{j-1}) \dots dx_{y-1} \end{pmatrix}, \tag{8}$$

where $y_m \equiv x$, and the ellipses indicate that the right-hand factor must be placed under the integral sign. The invariant point will be stable in the small if $|v^m(x)| < |v^0(x)|$ for all $x \in [a,b]$.

Verification of this condition can be reduced to a known procedure ([11]) by means of, say, one of the following devices:

1) In equations (8) take all quantities in modulus, and the functions of x by the maximum of the modulus and, moreover, replace

$$|\Delta u^j(l)| \to \max_{a<x<b} |\delta u^j(x)|.$$

As a result we obtain a difference system of two inequalities with constant coefficients.

2) Proceed analogously, but do not make the replacement; then from (8) we obtain a system of three inequalities of the same kind

$$\max_{a<x<b} |v^m(x)| \le \max_{a<x<b} \prod_{j=m}^{1} |A_j| \max_{a<x<b} |v^0(x)|. \tag{9}$$

Replacing, in cases 1) and 2), the inequalities by equalities and putting $\max_x |v^m| = z \max_x |v^0|$, in both cases we obtain, as the condition for the existence of a nontrivial solution, a quadratic equation for z (from (8) it follows that

$$\det\left(\max_x \prod_{j=m}^{1} |A_j|\right) = 0$$

). For stability of the invariant point it is sufficient that the roots of this equation lie inside the unit circle. Note that the cruder sufficient condition of method 1) gives, in the problem under consideration, for any m the stability region of zero dimension ($|\varphi'| \leq 0$).

3) Necessary and sufficient conditions for stability of the invariant point can be obtained if, for equations (8), one repeats the Fredholm procedure ([7–9]), i.e. divides the interval $a \leq x \leq b$ into small subintervals and transforms this system into an algebraic homogeneous one by introducing the equality $v^0(x_i) = \lambda v^m(x_i)$. Next, in setting the determinant of this system equal to zero, expand the latter with respect to the elements of the upper row and, carrying out the Fredholm limiting passage, obtain the characteristic equation for λ.

This very laborious path can be substantially simplified if one observes that it makes legitimate, for obtaining necessary and sufficient stability conditions, the substitution in equations (8)

$$\delta u^0(x) = \lambda \delta u^m(x),$$

in which λ does not depend on x.

After this the lower equation (8) turns into a Fredholm integral equation of the second kind with parameter λ. Solving it through the re-

the resolvent and substituting the result into the upper equations, one can obtain the system

$$\begin{pmatrix} \Delta\mu^m \\ \Delta u^m(l) \end{pmatrix} = \begin{pmatrix} a_{1m} & a_{1m} \\ a_{2m} & a_{2m} \end{pmatrix} \begin{pmatrix} \Delta\mu^0 \\ \Delta u^0(l) \end{pmatrix}, \tag{10}$$

where the quantities a_{1m} and a_{2m} are comparatively easy to compute. Setting in (10) $\Delta\mu^0 = \lambda\Delta\mu^m$ and $\Delta u^0(l) = \lambda\Delta u^m(l)$, we finally obtain the characteristic equation in the form

$$1 - \lambda(a_{1m} + a_{2m}) = 0. \tag{11}$$

For stability of the invariant point under consideration, the roots of this equation must be greater than unity in modulus. As an example, we give the values of the quantities a_{im} for $m = 1, 2$:

$$a_{11} = 1, \qquad a_{21} = \varphi_1' \left(f^1(l) + \lambda \int_a^b \Gamma_1(l, y; \lambda) f^1(y)\, dy \right); \tag{12}$$

$$a_{12} = 1 + \varphi_1' f^1(l) + \lambda \int_a^b K^1(l, y) a_{22}(y)\, dy, \qquad a_{22} = \alpha(l) + \lambda \int_a^b \Gamma_{1,2}(l, x; \lambda) \alpha(x)\, dx,$$

where

$$\alpha(y) = \varphi_2' f^1(y)(1 + \varphi_1' f^1(l)) + \varphi_1' f^2(y),$$

and $\Gamma_{1,2}(y, x; \lambda)$ is the resolvent of the kernel

$$K_{1,2}(y, x) = \varphi_2' f^1(y) K^1(l, x) + K^2(y, x).$$

In conclusion, we note that the procedure considered can be extended to the multidimensional case ([9]) and to the case of an unbounded domain ([7],[9]).

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Note: Figure translations are in progress. See original paper for figures.