THE SUSLIN NUMBER AND CARDINALITY, CHARACTERS OF POINTS IN SEQUENTIAL BICOMPACTS

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THE SUSLIN NUMBER AND CARDINALITY, CHARACTERS OF POINTS IN SEQUENTIAL BICOMPACTS

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The notation and definitions used here are explained in ([1],[2],[4–7]). We note specifically the following. Of topological spaces, only Hausdorff spaces are considered; of cardinal numbers, only infinite ones. The latter are denoted by the symbols λ, τ. The character of a set A in a space X is $\chi(A, X) = \min\{|B| : B$ is a base of A in $X\}$, where a base of A in X is a family of open sets containing arbitrarily small neighborhoods of A. If $x \in X$, then $\chi(x, X) = \chi(\{x\}, X)$. The character of a space X is $\chi(X) = \sup\{\chi(x, X) : x \in X\}$*. The combination (G.C.H.) means that the generalized continuum hypothesis is assumed to hold. By $|\alpha|$ is denoted the cardinality of the set well ordered by type α. If M is a set, then $\exp M = \{A : A \subset M\}$ and $\exp_\tau M = \{A : A \subset M$ and $|A| \leq \tau\}$.

Definition 1. The **height** $h(x, X)$ of a space X at a point x is

$$\min\{\chi(F, X) : x \in F \subset X,\ F \text{ is bicompact}\}.$$

Definition 2. The height $h(X)$ of a space X is $\sup\{h(x, X) : x \in X\}$.

Definition 3. The **bitightness** $bt(X)$ of a space X is the least of those τ such that if $M \subset X$ and $[M] \neq M$, then there exist $x \in X \setminus M$ and a family $\lambda \subset \exp_\tau M$ for which: a) $|\lambda| \leq \tau$ and b) $\{x\} = \cap\{[P] : P \in \lambda\}$.

Definition 4. The **divergence** $\operatorname{div}(X)$ of a space X is the least τ such that for every set M not closed in X there exist $x \in [M] \setminus M$ and a family $\xi \subset \exp_\tau M$ converging** to x, for which $|\xi| \leq \tau$.

Proposition 1. $t(X) \leq bt(X) \leq \operatorname{div}(X) \leq \chi(X)$***.

Proposition 2. If Y is a quotient space of a space X, then $\operatorname{div}(Y) \leq \operatorname{div}(X)$.

Lemma 1. If $A \subset X$, then $|[A]| \leq |A|^{bt(X)}$.

Proof. We shall assume that A is infinite. Put $bt(X) = \tau$. Define, by transfinite induction, for every ordinal number α less than τ^+, the set A_α as follows: $A_0 =$

A; if A_α is defined, then

$$A_{\alpha+1} = \{x \in X : \text{ there exists } \lambda \in \exp_\tau(\exp_\tau A), \text{ for which } \{x\} = \cap\{[P] : P \in \lambda\}\};$$

if β is a limit ordinal and the sets A_α are defined for all $\alpha < \beta$, then $A_\beta = \cup\{A_\alpha : \alpha < \beta\}$. We shall show that: I. $|A_\alpha| \leq |A|^\tau$ for all $\alpha < \tau^+$, and II. $\cup\{A_\alpha : \alpha < \tau^+\}$ is a closed set.

Suppose that relation I is not true, and let α_0 be the first transfinite ordinal for which it is violated. Then $\alpha_0 > 0$, since $|A_0| = |A| \leq |A|^\tau$. The transfinite ordinal α_0 cannot be a limit ordinal—in the contrary case $|A_{\alpha_0}| \leq$ $\leqslant \sum\{|A_\alpha| : \alpha < a_0\} \leqslant |A|^\tau$, since $|a_0| < \tau^+ \leqslant |A|^\tau$ and $|A_\alpha| \leqslant |A|^\tau$ for all $\alpha < a_0$ by the choice of a_0. Hence, $a_0 = \alpha' + 1$. But $|A_{\alpha'}| \leqslant |A|^\tau$. It is known that for any set B the cardinality of the family $\exp_\tau B$ does not exceed $|B|^\tau$ (^{9}). To each point $x \in A_{\alpha'+1}$ assign some family $\lambda(x)$ of subsets of the set $A_{\alpha'}$, for which $|\lambda(x)| \leqslant \tau$, $|\bigcup\{P : P \in \lambda(x)\}| \leqslant \tau$, and $\{x\} = \bigcap\{[P] : P \in \lambda(x)\}$. Obviously, this defines a one-to-one mapping of the set $A_{\alpha'+1}$ into the set $\exp_\tau(\exp_\tau A_{\alpha'})$. Hence,

$$|A_{\alpha'+1}| \leqslant |\exp_\tau(\exp_\tau A_{\alpha'})| \leqslant (|A_{\alpha'}|^\tau)^\tau \leqslant |A|^\tau.$$

Relation I is proved.

Now suppose that relation II is not true, i.e., that the set $C = \bigcup\{A_\alpha : \alpha < \tau^+\}$ is not closed. Then there exist a point $z \notin C$ and a family $\mu \in \exp_\tau(\exp_\tau C)$ such that $\{z\} = \bigcap\{[P] : P \in \mu\}$. Obviously, for some $a^* < \tau^+$, then $\bigcup\{P, P \in \mu\} \subset \bigcup\{A_\alpha : \alpha < a^*\} \subset A_{a^*}$. From the definition of the set A_{a^*+1} it now follows that $z \in A_{a^*+1}$, a contradiction. Thus, C is a closed set. Moreover, by I, $|C| \leqslant \tau^+|A|^\tau \leqslant |A|^\tau$. But $A = A_0 \subset C$; consequently, $[A] \subset C$ and $|[A]| \leqslant |A|^\tau$. Lemma 1 is proved.

We shall need the following two assertions:

VI (transitivity of character). If $X_2 \subset X_1 \subset X$, where X_2 and X_1 are bicompacts, then $\chi(X_2, X) \leqslant \chi(X_2, X_1) + \chi(X_1, X)$ (^{3}).

VII. If X is τ-compact, $bt(X) \leqslant \tau$, and $\chi(X) \leqslant 2^\tau$, then $|X| \leqslant 2^\tau$.

Assertion VII is obtained in an obvious way from the main theorem 2 of (^{1}) by means of Lemma 1.

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* If M is a set of cardinal numbers, then $\sup M$ is the least of the cardinal numbers not smaller than every element of M.

** A family ξ converges to a point x if every neighborhood of the point x contains some element of the family ξ.

*** It is easily proved that the weak tightness $t_c(X)$ and the tightness $t(X)$ in the sense of (^{2}) always coincide. Therefore $t(X) = t_c(X) \leq bt(X)$.

Lemma 2 (main). Let X be a bicompact, $t(x, X) < \tau \leqslant \lambda$ for every point $x \in X$, and let the cardinal number τ be regular. Suppose further that from $A \subset X$ and $|A| < \tau$ it follows, for every $A \subset X$, that $|[A]| < \lambda$. Then there exists a closed set B in X for which $\chi(B, X) < \tau$ and $0 < |B| < \lambda$.

Lemma 3. Every nonempty open subset of a regular space contains a nonempty closed-in-this-space set of type G_δ.

Proof of Lemma 2. If $|X| < \lambda$, the assertion is obvious. If $|X| \geqslant \lambda$, one can carry out a construction by transfinite induction, as a result of which to each $\alpha < \tau$ there will be assigned a point $x(\alpha)$ so that the set $\{x(\alpha) : \alpha < \tau\}$, well ordered in accordance with the order of the transfinite numbers, will be a free sequence of length τ (1). In parallel, nonempty closed sets $F(\alpha)$ will be defined.

As $x(0)$ choose any point of the set X. Take as $F(0)$ some nonempty set closed in X, of type G_δ, not containing the point $x(0)$. Let $\alpha_0 < \tau$, and suppose that for every $\alpha < \alpha_0$ a point $x(\alpha)$ and a nonempty closed set $F(\alpha) \subset X$ have already been defined so that $\chi(F(\alpha), X) \leqslant |\alpha| + \aleph_0$, and if $\alpha_1 < \alpha_2 < \alpha_0$, then $F(\alpha_1) \supset F(\alpha_2)$. Introduce the notation $A(\alpha_0) = \{x(\alpha) : \alpha < \alpha_0\}$. Then $|A(\alpha_0)| \leqslant |\{\alpha : \alpha < \alpha_0\}| < \tau$, and therefore $|[A(\alpha_0)]| < \lambda$. Put $B(\alpha_0) = \bigcap\{F(\alpha) : \alpha < \alpha_0\}$. If $|B(\alpha_0)| < \lambda$, then $B(\alpha_0)$ is the desired set, for $\{F(\alpha) : \alpha < \alpha_0\}$ is a centered family of nonempty, closed-in-X sets, X is a bicompact, the character and pseudocharacter of a closed set in a bicompact coincide, and $\sum\{|\alpha| : \alpha < \alpha_0\} \leqslant |\alpha_0| + \aleph_0 < \tau$. If $|B(\alpha_0)| \geqslant \lambda$, then $B(\alpha_0) \setminus [A(\alpha_0)] \neq \Lambda$. Take as $x(\alpha_0)$ any point of the set $B(\alpha_0) \setminus [A(\alpha_0)]$. Define $F(\alpha_0)$ as some nonempty set, closed in X, contained in $B(\alpha_0) \setminus ([A(\alpha_0)] \cup \{x(\alpha_0)\})$, of type G_δ in $B(\alpha_0)$. By VI,

$$\chi(F(\alpha_0), X) \leqslant \chi(B(\alpha_0), X) + \aleph_0 \leqslant |\alpha_0| + \aleph_0.$$

We shall show that the described process of defining the points $x(\alpha)$ and the sets $F(\alpha)$, under any realization of it, stops at some transfinite ordinal smaller than τ. The realization determines only which particular transfinite ordinal $\alpha^* < \tau$ turns out to be obstructing. For this α^* we then have $|B(\alpha^*)| < \lambda$ and $\chi(B(\alpha^*), X) \leqslant |\alpha| + \aleph_0 < \tau$, whence it follows that $B(\alpha^*)$ is the desired set.

···set. Suppose, however, that the sets $F(\alpha)$ and the points $x(\alpha)$ have been defined for all $\alpha < \tau$. Put $A = \{x(\alpha) : \alpha < \tau\}$, and $x(\alpha_1) < x(\alpha_2)$ if and only if $\alpha_1 < \alpha_2$. For $\alpha_1 \neq \alpha_2$, $x(\alpha_1) \neq x(\alpha_2)$. Hence $|A| = \tau$. By the construction, if $\alpha_0 < \tau$, then $[\{x(\alpha) : \alpha \leq \alpha_0\}] \cap F(\alpha_0) = \Lambda$, and if $\beta > \alpha_0$, then $x(\beta) \in F(\alpha_0)$. Therefore

$$[\{x(\alpha) : \alpha \leq \alpha_0\}] \cap \{x(\beta) : \alpha_0 < \beta\} \subset [\{x(\alpha) : \alpha \leq \alpha_0\}] \cap F(\alpha_0) = \Lambda.$$

Thus it has been proved that $A, <$ is a free sequence of length τ. But then some point $y \in X$ is a point of complete accumulation for A. Now a simple fact is needed (see (1)).

Lemma 4. *If the tightness at every point of a topological space X is less than τ, and τ is a regular cardinal number, then in X there is no point of complete accumulation for any free sequence of length τ.*

Lemma 5. *Suppose that all the assumptions of Lemma 2 are satisfied. Then there exists a family γ of pairwise disjoint closed subsets of X such that $[\bigcup\{P : P \in \gamma\}] = X$, and for every $P \in \gamma$, $\chi(P, X) < \tau$ and $0 < |P| < \lambda$.*

Lemma 5 is easily proved on the basis of the Kuratowski-Zorn principle.

Definition 5. A family $\mathcal{E} = \{\gamma_\alpha : \alpha \in M\}$ of families of subsets of a set X is called **Hausdorff** if, for any $\alpha_1, \alpha_2 \in M$, where $\alpha_1 \neq \alpha_2$, there exist $P_1 \in \gamma_{\alpha_1}$ and $P_2 \in \gamma_{\alpha_2}$ such that $P_1 \cap P_2 = \Lambda$. A family γ of sets is called a **prefilter** if from $P' \in \gamma$, $P'' \in \gamma$ it follows that there exists $P \in \gamma$ for which $P \subset P' \cap P''$.

Lemma 6. *Let $\mathcal{E} = \{\gamma_\alpha : \alpha \in M\}$ be a Hausdorff family of prefilters in X, with $|M| > 2^\tau$ and $|\gamma_\alpha| \leq \tau$ for every $\alpha \in M$. Then there exist a set $M' \subset M$ and sets $P_\alpha \in \gamma_\alpha$ for every $\alpha \in M'$, such that $|M'| > \tau$ and the family $\{P_\alpha : \alpha \in M'\}$ is disjoint.*

Lemma 6 easily follows from Lemma 6 of paper ([4]). Lemmas 7 and 8 are obvious.

Lemma 7. *If $x \in U \subset X$, where U is open in X, and $h(x, X) \leq \tau$, then there exists a bicompact $\Phi \subset X$ for which $x \in \Phi \subset U$ and $\chi(\Phi, X) \leq \tau$.*

Lemma 8. *If $h(X) \leq \tau$, then there exists a disjoint family S of bicompact subsets of the space X such that $(j_1) : [\bigcup\{F : F \in S\}] = X$, and $(j_2) : \chi(F, X) \leq \tau$ for every $F \in S$.*

Lemma 9. *If X is bicompact and $bt(X) \leq \tau$, then there exists a disjoint family γ of closed subsets of X such that: $(k_1) : [\bigcup\{P : P \in \gamma\}] = X$; (k_2) : if $P \in \gamma$, then $|P| \leq 2^\tau$, and $(k_3) : \chi(P, X) \leq \tau$ for all $P \in \gamma$.*

Proof. All the assumptions of Lemma 2 are satisfied with respect to the space X and the cardinal numbers τ^+ in the role of τ and $(2^\tau)^+$ in the role of λ, since τ^+ is regular, $t(X) \leq bt(X) < \tau^+$, and if $A \subset X$, $|A| < \tau^+$, then $|A| \leq \tau$ and $|[A]| \leq \tau^\tau = 2^\tau < (2^\tau)^+ = \lambda$ by Lemma 1. Therefore Lemma 5 is applicable. Lemma 9 is proved.

Lemma 10. *If F is closed in X, then $bt(F) \leq bt(X)$.*

Theorem 1. $|X| \leq 2^{bt(X)+c(X)+h(X)}$.

Proof. Choose a family S of nonempty subsets of the space X in accordance with Lemma 8. To each $F \in S$ apply Lemma 9—in accordance with it choose a family γ_F of nonempty subsets of the space F. Put $\mathcal{E} = \{\gamma_F : F \in S\}$. By the transitivity of character (VI), $\chi(\Phi, X) \leq bt(X) + h(X)$ for every $\Phi \in \mathcal{E}$. To each $\Phi \in \mathcal{E}$ assign some base ξ_Φ of the set Φ in X, for which $|\xi_\Phi| \leq bt(X) + h(X)$. Since X is a Hausdorff space, and the family $\mathcal{E}$ consists of pairwise disjoint nonempty bicompacts, $\{\xi_\Phi : \Phi \in \mathcal{E}\}$ is a Hausdorff family of prefilters. If we had

$$|\mathcal{E}| > 2^{c(X)+bt(X)+h(X)},$$

then, by Lemma 6, there would be a disjoint family of elements of these prefilters whose cardinality is greater than $c(X)$, and this is impossible, since the prefilters ξ_Φ, $\Phi \in \mathcal{E}$, consist of open sets. Hence

$$|\mathcal{E}| \leq 2^{c(X)+bt(X)+h(X)}.$$

From this and from condition (k_2) of Lemma 9 it follows that

$$|\bigcup\{\Phi : \Phi \in \mathcal{E}\}| \leq 2^{c(X)+bt(X)+h(X)}.$$

But, obviously, $[\bigcup\{\Phi : \Phi \in \mathcal{E}\}] = X$. By Lemma 1 it follows that

$$|[\bigcup\{\Phi : \Phi \in \mathcal{E}\}]| \leq |\bigcup\{\Phi : \Phi \in \mathcal{E}\}|^{bt(X)} \leq \left(2^{bt(X)+c(X)+h(X)}\right)^{bt(X)} = 2^{bt(X)+c(X)+h(X)}.$$

Theorem 1 is proved.

Theorem 2. $[\{x \in X : \chi(x, X) \leq 2^{bt(X)} + h(X)\}] = X$.

Proof. We apply Lemmas 8 and 9, in particular using condition (k_2) from the formulation of Lemma 9 (note that if F is compact, then $\chi(x, F) \leq |F|$); we shall further refer to the transitivity of character (VI).

Theorem 3 (G.C.H.). $[\{x \in X : \chi(x, X) \leq bt(X) + h(X)\}] = X$.

Proof. Choose in X (Lemmas 8 and 9) a disjoint family $\mathcal{E}$ of compacta whose character in X does not exceed $h(X) + bt(X)$, and whose cardinality does not exceed $2^{bt(X)}$, such that $[\bigcup\{\Phi : \Phi \in \mathcal{E}\}] = X$. It is known (8) that if X is a compactum and $\chi(x, X) > \tau$ for all $x \in X$, then $|X| \geq 2^{(\tau+)}$. Assuming that $2^{(\tau+)} > 2^\tau$ (this follows from the generalized continuum hypothesis), we conclude that in each $\Phi \in \mathcal{E}$ the points whose character in Φ does not exceed $bt(X)$ form an everywhere dense subset. The conclusion of the theorem now follows from the transitivity of character.

A topological space is called homogeneous if for any $x, y \in X$ there exists a homeomorphism $f : X \to X$ such that $f(x) = y$ and $f(X) = X$.

Theorem 4. If X is homogeneous, then $\chi(X) \leq 2^{bt(X)} + h(X)$.

Theorem 5 (G.C.H.). If X is homogeneous, then $\chi(X) \leq bt(X) + h(X)$.

Theorem 6. If X is homogeneous, then $|X| \leq 2^{ic(X)+bt(X)+h(X)}$ (see (1)).

Theorem 4 follows from Theorem 2, Theorem 5 follows from Theorem 3, and Theorem 6 follows from Theorem 4 and (1) (see Theorem 2).

1st group of corollaries. Let X be a space of point-countable type. Then:
a) if X is sequential and satisfies Suslin' s condition, then $|X| \leq 2^{\aleph_0}$;
b) if X is sequential, then $[\{x \in X : \chi(x, X) \leq 2^{\aleph_0}\}] = X$;
c) (C.H.) if X is sequential, then $[\{x \in X : \chi(x, X) \leq \aleph_0\}] = X$;
d) $[\{x \in X : \chi(x, X) \leq 2^{bt(X)}\}] = X = [\{x \in X : \chi(x, X) \leq 2(2^{t(x)})\}]$;
e) (G.C.H.) $[\{x \in X : \chi(x, X) \leq bt(X)\}] = X = [\{x \in X : \chi(x, X) \leq 2^{t(X)}\}]$;

f) (C.H.) if X is a quotient space of a space with the first axiom of countability, then $[\{x \in X : \chi(x, X) \leq \aleph_0\}] = X$.

2nd group of corollaries. Let X be a homogeneous compactum. Then:
a) if the space X is sequential, then $|X| \leq 2^{\aleph_0}$;
b) (C.H.) if the space X is sequential, then $\chi(X) \leq \aleph_0$;
c) $|X| \leq 2^{bt(X)}$;
d) (G.C.H.): $\chi(X) \leq bt(X)$.

Note added in proof. The following holds.

Theorem 7 (G.C.H.). The cardinality of a homogeneous compactum cannot be a limit cardinal number.

From the proofs of Lemmas 2, 5 and Lemmas 8, 9 there evidently follows

Theorem 8. If $h(X) \leq \aleph_0$ and $c(X') \leq \aleph_0$ for all $X' \subset X$, then $X = [X^*]$, where $|X^*| \leq 2^{\aleph_0}$.

An analogue of this theorem is true for any τ.

The idea of formulating Theorem 8 was suggested to me by B. Shapirovskii after he had read an oral presentation of this work.

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Note: Figure translations are in progress. See original paper for figures.