EQUATIONS OF NONLINEAR ELASTICITY IN DISPLACEMENTS

THEORY OF ELASTICITY

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Abstract

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THEORY OF ELASTICITY

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EQUATIONS OF NONLINEAR ELASTICITY IN DISPLACEMENTS

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1. As strain tensors, along with the Green and Almansi tensors

$$\overset{\circ}{\mathbf{E}} = \tfrac{1}{2}(\ \cdot\ ^{*} - \overset{\circ}{\mathbf{I}}) = \tfrac{1}{2}(\overset{\circ}{\nabla}\mathbf{u} + \mathbf{u}\overset{\circ}{\nabla} + \overset{\circ}{\nabla}\mathbf{u}\cdot\mathbf{u}\overset{\circ}{\nabla}), \qquad \overset{\circ}{\mathbf{I}} = \overset{\circ}{\mathfrak{e}}{}^{i}\overset{\circ}{\mathfrak{e}}_{i}; \tag{1}$$

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{I} - \ ^{-1}\cdot\ ^{*-1}) = \tfrac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla - \nabla\mathbf{u}\cdot\mathbf{u}\nabla), \qquad \mathbf{I} = \mathfrak{e}^{i}\mathfrak{e}_{i}, \tag{2}$$

one also considers Hencky tensors, defined by the formulas ([1])

$$\overset{\circ}{\mathbf{H}} = \tfrac{1}{2}\ln(\overset{\circ}{\mathbf{I}} + 2\overset{\circ}{\mathbf{E}}), \qquad \mathbf{H} = -\tfrac{1}{2}\ln(\mathbf{I} - 2\mathbf{E}). \tag{3}$$

The tensors (1) and (2) are related by the relations

$$\mathbf{E} = \ ^{-1}\cdot\overset{\circ}{\mathbf{E}}\cdot\ , \qquad \overset{\circ}{\mathbf{E}} = \ \cdot\mathbf{E}\cdot\ ^{*}. \tag{4}$$

Here, by $\ = \overset{\circ}{\mathbf{I}} + \overset{\circ}{\nabla}\mathbf{u} = (\mathbf{I} - \nabla\mathbf{u})^{-1}$ is denoted the deformation affinor, establishing the correspondence between the set of linear elements $d\overset{\circ}{\mathbf{r}}$ and $d\mathbf{r} = d\overset{\circ}{\mathbf{r}} + d\mathbf{u}$ in a neighborhood of an arbitrary point of the body in the initial and deformed states ([2]). The asterisk denotes the operation of conjugation.

Differentiation in (1) is carried out with respect to the initial (Lagrangian) coordinates ξ^i, and in (2) with respect to the coordinates η^i of the reference system

$$\overset{\circ}{\nabla}\mathbf{u} = \overset{\circ}{\mathfrak{e}}{}^{i}\,\partial\mathbf{u}/\partial\xi^{i}, \qquad \nabla\mathbf{u} = \mathfrak{e}^{i}\partial\mathbf{u}/\partial\eta^{i}, \qquad \overset{\circ}{\mathfrak{e}}_{i} = \partial\overset{\circ}{\mathbf{r}}/\partial\xi^{i}, \qquad \mathfrak{e}_{i} = \partial\mathbf{r}/\partial\eta^{i}.$$

Below we shall need variations of the tensors (1) and (2) in the form

$$\ ^{-1}\cdot\delta\overset{\circ}{\mathbf{E}}\cdot\ ^{*-1} = \nabla(\delta\mathbf{r}) + (\delta\mathbf{r})\nabla, \tag{5}$$

$$\delta\mathbf{E} = \tfrac{1}{2}\left[\nabla(\delta\mathbf{r})\cdot(\mathbf{I}-2\mathbf{E}) + (\mathbf{I}-2\mathbf{E})\cdot(\delta\mathbf{r})\nabla\right]. \tag{6}$$

2. The stressed state of an individual particle will be characterized by symmetric tensors of true

$$\mathbf{T} = T^{ij}\mathfrak{e}_i\mathfrak{e}_j = \hat{T}^{ij}\hat{\mathfrak{e}}_i\hat{\mathfrak{e}}_j$$

and generalized

$$\mathring{\Sigma} = (1+\Delta)\hat{T}^{ij}\mathring{\mathfrak{e}}_i\mathring{\mathfrak{e}}_j$$

stresses, related to one another by the formulas

$$\Sigma = (1+\Delta)\mathbf{T} = {}^{*}\cdot\mathring{\Sigma}\cdot\ , \qquad \mathring{\Sigma} = {}^{*-1}\cdot\Sigma\cdot\ {}^{-1}, \qquad 1+\Delta = \exp(\mathbf{H}:\mathbf{I}). \tag{7}$$

By $\hat{\mathfrak{e}}_i$ is denoted the accompanying basis ([1]).

The equations of motion of a particle in the reference system have the form ([2])

$$\nabla\cdot\mathbf{T} + \rho\mathbf{F} = \rho\mathbf{a}, \tag{8}$$

where $\mathbf{a}$ is acceleration, $\mathbf{F}$ body forces, ρ density.

In the basis $\mathring{\mathfrak{e}}_i$ the equations of motion are written in the form ([3,4])

$$\mathring{\nabla}\cdot\mathring{\Sigma} + \mathring{\Sigma} : (\mathring{\nabla}\ \cdot\ {}^{-1}) + \rho_0\mathbf{F}\cdot\ {}^{-1} = \rho_0\ddot{\mathbf{u}}\cdot\ {}^{-1}, \qquad \rho_0 = \rho(1+\Delta). \tag{9}$$

The boundary conditions for the true stress tensor have the form

$$\mathbf{T}\cdot\mathbf{n} = \mathbf{P}_n, \tag{10}$$

where $\mathbf{n} = n^i\mathbf{e}_i$ is the unit normal to the surface of the body in the deformed state; $\mathbf{P}_n$ is the vector of external surface forces. The boundary condition for the tensor $\mathring{\Sigma}$ can be obtained by transforming the normals to an individual area element when it passes from the initial to the deformed state ([2]), $\mathbf{n}\,d\sigma = (1+\Delta)\Phi^{-1}\cdot\mathbf{n}\,d\mathring{\sigma}$.

From (10), using the last formula and the equalities (7), the stated boundary condition follows:

$$\mathring{\Sigma}\cdot\mathbf{n} = \mathring{\mathbf{P}}_n\cdot\Phi^{-1} = \mathring{\mathbf{P}}_n\cdot(\mathbf{I}+\mathring{\nabla}\mathbf{u})^{-1}, \qquad \mathring{\mathbf{P}}_n = \mathbf{P}_n\,d\sigma/d\mathring{\sigma}, \tag{11}$$

where $\overset{\circ}{\mathbf{P}}_n$, as follows from the last formulas, is the stress vector acting in the final state on the area element $\mathbf{n}\,d\sigma$ and referred to unit area of the same element in the undeformed state.

3. In correspondence with the two kinds of stress tensors, the elementary work of these stresses on variations $\delta\mathbf{u} = \delta\mathbf{r}$ of the displacement vector of an individual particle can be represented as follows:

$$\delta A = \Sigma : {}^1\!/_2[\nabla(\delta\mathbf{r}) + (\delta\mathbf{r})\nabla], \qquad \delta A = \overset{\circ}{\Sigma} : \delta\overset{\circ}{\mathbf{E}}. \tag{12}$$

The right-hand sides of these formulas are reduced one to the other by using the equalities (5), (7).

Let us postulate, as is usually done in nonlinear elasticity theory ([4,5]), that the elementary work is the variation of a stress potential depending on the invariants of one of the tensors $\mathbf{H}, \mathbf{E}, \overset{\circ}{\mathbf{E}}, \overset{\circ}{\mathbf{H}}$ and existing, at least, in adiabatic and isothermal processes.

Using the invariants of the tensor

$$E_1 = \mathbf{I} : \mathbf{E}, \quad E_2 = \mathbf{I} : \mathbf{E}^2, \quad E_3 = \mathbf{I} : \mathbf{E}^3, \dots, \quad E_n = \mathbf{I} : \mathbf{E}^n, \quad \mathbf{E}^3 = \mathbf{E} \cdot \mathbf{E} \cdot \mathbf{E} \tag{13}$$

as an example, introduce a symbolic differential operator from scalar functions of these invariants with respect to the tensor argument $\mathbf{E}$. Varying (13), we obtain

$$\delta E_n = \delta(\mathbf{I} : \mathbf{E}^n) = n\mathbf{E}^{n-1} : \delta\mathbf{E} = \frac{\partial(\mathbf{I} : \mathbf{E}^n)}{\partial\mathbf{E}} : \delta\mathbf{E}.$$

Here the notation of the symbolic derivative has been introduced,

$$\frac{\partial(\mathbf{I} : \mathbf{E}^n)}{\partial\mathbf{E}} = n\mathbf{E}^{n-1},$$

which is analogous to the derivative of a power scalar function.

With the aid of the symbolic derivative, the variation of the potential is written as follows:

$$\delta A = \frac{\partial A}{\partial\mathbf{E}} : \delta\mathbf{E} = \frac{\partial A}{\partial\mathbf{H}} : \delta\mathbf{H}, \qquad \delta A = \frac{\partial A}{\partial\overset{\circ}{\mathbf{E}}} : \delta\overset{\circ}{\mathbf{E}}. \tag{14}$$

As a result of comparing the last equalities from (12) and (14), we obtain

$$\overset{\circ}{\Sigma} = \frac{\partial A}{\partial \overset{\circ}{\mathbf{E}}}. \tag{15}$$

Substituting into the first equality of (14) first the variation (6) of the tensor **E**, and then bringing under the sign of variation of the same equality the value

$$\mathbf{E} = \tfrac{1}{2}[\mathbf{I} - \exp(-2\mathbf{H})],$$

we obtain

$$\delta A = \frac{\partial A}{\partial \mathbf{E}} \cdot (\mathbf{I} - 2\mathbf{E}) : {}^{1}\!/_{2}[\nabla(\delta\mathbf{r}) + (\delta\mathbf{r})\nabla] = \frac{\partial A}{\partial \mathbf{E}} \cdot (\mathbf{I} - 2\mathbf{E}) : \delta\mathbf{H}. \tag{16}$$

From comparison of (12), (14), and (16) it follows that

$$\Sigma = \frac{\partial A}{\partial \mathbf{E}} \cdot (\mathbf{I} - 2\mathbf{E}) = \frac{\partial A}{\partial \mathbf{H}}. \tag{17}$$

Now the elementary work can be represented in the form ([3], [4])

$$\delta A = \Sigma : \delta\mathbf{H} = \frac{1}{3}\sigma_0 \delta h_0 + 3\sigma(\cos\omega\, \delta h + h \sin\omega\, \delta\beta), \quad \omega = \varphi - \beta. \tag{18}$$

$$\omega = \varphi - \beta.$$

Here Σ_∂, $\mathbf{H}_\partial$, ω are the deviators and the phase difference of the tensors Σ and $\mathbf{H}$; σ_0, σ and h_0, h are the normal and tangential components of the tensor Σ on the octahedral plane and the analogous components of the tensor **H**.

In formula (18), the first term $\frac{1}{3}\sigma_0\delta h_0$ represents the elementary work of volume change, while the second term is, essentially, the work of change of shape.

In formulating the physical laws we shall restrict ourselves to the case when the phases of the stresses and strains coincide and when the generalized law of volumetric deformation $\sigma_0 = \sigma_0(h_0)$ and the generalized law of change of shape $\sigma = \sigma(h)$ are satisfied. We shall approximate these experimental laws by analytic functions

$$\sigma_0 = 3K \sum_s k_s h_0^s, \qquad \sigma = 2G \sum_s g_s h^s, \qquad k_1 = g_1 = 1.$$

The potential corresponding to these laws has the form

$$A = K \sum_s \frac{k_s}{s+1} h_0^{s+1} + 6G \sum_s \frac{g_s}{s+1} h^{s+1}. \tag{19}$$

Hence, by formula (17), we find

$$\Sigma = K\left(\sum_s k_s h_0^s\right)\mathbf{I} + 2G\left(\sum_s g_s h^{s-1}\right)\left(\mathbf{H} - \frac{1}{3}h_0\mathbf{I}\right). \qquad (20)$$

These relations generalize Kauderer' s physically nonlinear law ([6]) to the case of finite deformations ([7]). Setting in (20) $k_2 = g_2 = 0$ and discarding terms of the fourth order with respect to the deformations, we transform formula (20) to the variables $\mathbf{T}$, $\mathbf{E}$

$$\mathbf{T} = \lambda E_1\mathbf{I} + 2G\mathbf{E} + \lambda(E_2 - E_1^2)\mathbf{I} - 2GE_1\mathbf{E} + 2G\mathbf{E}^2 + [(Kk_3 + \tfrac{2}{27}Gg_3 +$$

$$+\tfrac{1}{2}\lambda + \tfrac{4}{9}G)E_1^3 - (2K + \tfrac{2}{9}Gg_3)E_1E_2 + \tfrac{4}{3}KE_3]\mathbf{I} - 2G[\tfrac{1}{3}(1-g_3)E_2 +$$

$$+(\tfrac{1}{6} + \tfrac{1}{9}g_3)E_1^2]\mathbf{E} - \tfrac{2}{3}GE_1\mathbf{E}^2, \qquad \lambda = K - \tfrac{2}{3}G. \qquad (21)$$

With the same degree of accuracy, if beforehand in the potential (19) one passes, by means of the invariants of the tensor $\mathring{\mathbf{H}}$, to the invariants of the tensor $\mathring{\mathbf{E}}$ and uses formula (15), one can obtain

$$\mathring{\Sigma} = \lambda\mathring{E}_1\mathring{\mathbf{I}} + 2G\mathring{\mathbf{E}} - \lambda\mathring{E}_2'\mathring{\mathbf{I}} - 2\lambda\mathring{E}_1\mathring{\mathbf{E}} + \qquad (22)$$

$$+[(Kk_3 + \tfrac{2}{27}Gg_3 + \tfrac{22}{3}G)\mathring{E}_1^3 - (\tfrac{22}{3} + \tfrac{2}{9}g_3)G\mathring{E}_1\mathring{E}_2 + (\tfrac{4}{3}\lambda + \tfrac{44}{9}G)\mathring{E}_3]\mathring{\mathbf{I}} +$$

$$+[(2\lambda + \tfrac{22}{3}G + \tfrac{2}{9}Gg_3)\mathring{E}_2 - (\tfrac{22}{3} + \tfrac{2}{9}g_3)G\mathring{E}_1^2]\mathring{\mathbf{E}} + (4\lambda + \tfrac{44}{3}G)\mathring{E}_1\mathring{\mathbf{E}}^2.$$

Finally, passing to gradients of the displacement vector, we shall have

$$\mathbf{T} = \lambda\nabla\cdot\mathbf{u}\,\mathbf{I} + G(\nabla\mathbf{u} + \mathbf{u}\nabla) + \mathbf{T}_1, \qquad (23)$$

$$\begin{aligned}\mathbf{T}_1 = {}& \tfrac{1}{2}\lambda[\nabla\mathbf{u} : \mathbf{u}\nabla - 2(\nabla\cdot\mathbf{u})^2]\mathbf{I} - G\nabla\cdot\mathbf{u}(\nabla\mathbf{u} + \mathbf{u}\nabla) - G\nabla\mathbf{u}\cdot\mathbf{u}\nabla \\ & + \tfrac{1}{2}G(\nabla\mathbf{u}^2 + \nabla\mathbf{u}\cdot\mathbf{u}\nabla + \mathbf{u}\nabla\cdot\nabla\mathbf{u} + \mathbf{u}\nabla^2) + [\lambda\nabla\cdot\mathbf{u}\,\nabla\mathbf{u} : \mathbf{u}\nabla \\ & + \tfrac{1}{3}G(\nabla\mathbf{u}^2 : \mathbf{u}\nabla + \nabla\mathbf{u} : \mathbf{u}\nabla^2) + \tfrac{1}{6}K(\nabla\mathbf{u}^2 : \nabla\mathbf{u} + \mathbf{u}\nabla : \mathbf{u}\nabla^2) \\ & + (Kk_3 + \tfrac{2}{27}Gg_3 + \tfrac{1}{2}\lambda + \tfrac{4}{9}G)\cdot(\nabla\cdot\mathbf{u})^3 - (K + \tfrac{1}{9}Gg_3)\nabla\cdot\mathbf{u}(\nabla\mathbf{u} : \nabla\mathbf{u} \\ & + \nabla\mathbf{u} : \mathbf{u}\nabla)]\mathbf{I} + G[\nabla\cdot\mathbf{u}\,\nabla\mathbf{u}\cdot\mathbf{u}\nabla + \tfrac{1}{2}\nabla\mathbf{u} : \mathbf{u}\nabla(\nabla\mathbf{u} + \mathbf{u}\nabla)] \\ & - \tfrac{1}{2}G[(\nabla\mathbf{u} + \mathbf{u}\nabla)\cdot\nabla\mathbf{u}\cdot\mathbf{u}\nabla + \nabla\mathbf{u}\cdot\mathbf{u}\nabla\cdot(\nabla\mathbf{u} + \mathbf{u}\nabla)] \\ & - G[\tfrac{1}{6}(1 - g_3)(\nabla\mathbf{u} : \nabla\mathbf{u} + \nabla\mathbf{u} : \mathbf{u}\nabla) + (\tfrac{1}{6} + \tfrac{1}{9}g_3)(\nabla\cdot\mathbf{u})^2](\nabla\mathbf{u} + \mathbf{u}\nabla) \\ & + \tfrac{3}{2}G\nabla\cdot\mathbf{u}(\nabla\mathbf{u} + \mathbf{u}\nabla)^2\end{aligned}$$

for law (21), and analogously for law (22)

$$\Sigma = \lambda\nabla\cdot\mathbf{u}\,\mathbf{I} + G(\nabla\mathbf{u} + \mathbf{u}\nabla) + \Sigma_1, \tag{24}$$

where Σ_1 is not written out for lack of space. Here, for convenience, the circles over letters and symbols have been omitted.

Substituting (23) into (8), we obtain the Lamé equations in the reference system with basis $\mathfrak{e}_i$

$$(\lambda + G)\nabla(\nabla\cdot\mathbf{u}) - G\nabla\cdot\nabla\mathbf{u} = \mathbf{w}_1, \qquad \mathbf{w}_1 = \rho(\mathbf{a} - \mathbf{F}) - \nabla\cdot\mathbf{T}_1, \tag{25}$$

and from (9) and (24) we obtain the Lamé equations in the basis $\mathfrak{e}_i$

$$(\lambda + G)\nabla(\nabla\cdot\mathbf{u}) - G\nabla\cdot\nabla\mathbf{u} = \mathbf{w}_2, \tag{26}$$

$$\begin{aligned}\mathbf{w}_2 = {}& \rho_0(\ddot{\mathbf{u}} - \mathbf{F})\cdot(\mathbf{I} - \nabla\mathbf{u} + \nabla\mathbf{u}^2) - \nabla\cdot\Sigma_1 + \big[{}^1/2\lambda(\nabla\mathbf{u} : \nabla\mathbf{u} + \nabla\mathbf{u} : \mathbf{u}\nabla)\mathbf{I} \\ & - \lambda\nabla\cdot\mathbf{u}(\nabla\mathbf{u} + \mathbf{u}\nabla) + {}^3/2G(\nabla\mathbf{u} + \mathbf{u}\nabla)^2\big] : \nabla\nabla\mathbf{u} \\ & - \big[\lambda\nabla\cdot\mathbf{u}\,\mathbf{I} + G(\nabla\mathbf{u} + \mathbf{u}\nabla)\big] : (\nabla\nabla\mathbf{u} - \nabla\nabla\mathbf{u}\cdot\nabla\mathbf{u}).\end{aligned}$$

To solve equations (25), (26) we shall use the method of a small parameter. In this case all characteristics of the stress-strain state are represented by series, for example,

$$\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)} + \mathbf{u}^{(2)} + \dots, \qquad \mathbf{T} = \mathbf{T}^{(0)} + \mathbf{T}^{(1)} + \mathbf{T}^{(2)} + \dots$$

The first term (the zeroth approximation) of the series is the classical expression of the corresponding characteristic, while the subsequent approximations take into account corrections for nonlinearity.

The Lamé equations for the static case in the reference system in successive approximations have the form

$$(\lambda + K)\nabla(\nabla \cdot \mathbf{u}^{(k)}) + G\nabla \cdot \nabla \mathbf{u}^{(k)} = -\nabla \cdot \mathbf{T}_1^{(k)}, \qquad k = 0, 1, 2, \dots,$$

$$\begin{aligned}
\mathbf{T}_1^{(0)} &= 0, \quad \mathbf{T}_1^{(1)} = {}^1/2\lambda[\nabla\mathbf{u}^{(0)} : \nabla\mathbf{u}^{(0)} - 2(\nabla \cdot \mathbf{u}^{(0)})^2]\mathbf{I} - G\nabla\mathbf{u}^{(0)} : \mathbf{u}^{(0)}\nabla \\
&\quad - G\nabla \cdot \mathbf{u}^{(0)}(\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla) + {}^1/2G(\nabla\mathbf{u}^{(0)2} + \nabla\mathbf{u}^{(0)} \cdot \mathbf{u}^{(0)}\nabla + \mathbf{u}^{(0)}\nabla \cdot \nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla^2), \\
\mathbf{T}_1^{(2)} &= \lambda[\nabla\mathbf{u}^{(0)} : \nabla\mathbf{u}^{(1)} - 2\nabla \cdot \mathbf{u}^{(0)}\nabla \cdot \mathbf{u}^{(1)}]\mathbf{I} - G(\nabla\mathbf{u}^{(1)} \cdot \mathbf{u}^{(0)}\nabla + \nabla\mathbf{u}^{(0)} \cdot \mathbf{u}^{(1)}\nabla) \\
&\quad - G[\nabla \cdot \mathbf{u}^{(1)}(\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla) + \nabla \cdot \mathbf{u}^{(0)}(\nabla\mathbf{u}^{(1)} + \mathbf{u}^{(1)}\nabla)] \\
&\quad + {}^1/2G[(\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla) \cdot (\nabla\mathbf{u}^{(1)} + \mathbf{u}^{(1)}\nabla) + (\nabla\mathbf{u}^{(1)} + \mathbf{u}^{(1)}\nabla) \cdot (\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla)] \\
&\quad + [\lambda\nabla \cdot \mathbf{u}^{(0)}\nabla\mathbf{u}^{(0)} : \mathbf{u}^{(0)}\nabla + {}^1/3G(\nabla\mathbf{u}^{(0)2} : \mathbf{u}^{(0)}\nabla + \nabla\mathbf{u}^{(0)} : \mathbf{u}^{(0)}\nabla^2) \\
&\quad + {}^1/6K((\nabla\mathbf{u}^{(0)} : \nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla : \mathbf{u}^{(0)}\nabla^2) + (Kk_3 + {}^1/27Gg_3 + {}^1/2\lambda \\
&\qquad + {}^1/9G)(\nabla \cdot \mathbf{u}^{(0)})^3 - (K + {}^1/9Gg_3)\nabla \cdot \mathbf{u}^{(0)}(\nabla\mathbf{u}^{(0)} : \nabla\mathbf{u}^{(0)} + \nabla\mathbf{u}^{(0)} : \mathbf{u}^{(0)}\nabla)]\mathbf{I} \\
&\quad + G[\nabla \cdot \mathbf{u}^{(0)}\nabla\mathbf{u}^{(0)} \cdot \mathbf{u}^{(0)}\nabla + {}^1/2\nabla\mathbf{u}^{(0)} : \mathbf{u}^{(0)}\nabla(\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla)] \\
&\quad + G[{}^1/2(\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla) \cdot \nabla\mathbf{u}^{(0)} \cdot \mathbf{u}^{(0)}\nabla + {}^1/2\nabla\mathbf{u}^{(0)} \cdot \mathbf{u}^{(0)}\nabla \cdot (\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla)] \\
&\quad + {}^1/6G\nabla \cdot \mathbf{u}^{(0)}(\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla)^2 - {}^1/6G[(1 - g_3)(\nabla\mathbf{u}^{(0)} : \nabla\mathbf{u}^{(0)} + \nabla\mathbf{u}^{(0)} : \mathbf{u}^{(0)}\nabla) \\
&\qquad + (1 + {}^2/3g_3)(\nabla \cdot \mathbf{u}^{(0)})^2](\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\nabla).
\end{aligned}$$

The first approximation, following the classical one, takes into account geometric nonlinearity, and the second also physical nonlinearity through the constants k_3, g_3.

Example. In accordance with the two systems of Lamé equations in the bases $\mathfrak{e}_i$ and $\mathfrak{e}_i$, two types of problems may be solved. In problems of the first type the shape of the boundary in the initial state is regarded as known and, in particular, one may find the shape of the boundary in the deformed state. In static problems of the second type the shape of the boundary in the deformed state is prescribed and, in particular, the shape of the boundary in the initial state is determined.

As an example let us consider the problem of the stressed state of a body with a spherical cavity in the deformed state and with a prescribed constant stress p at infinity.

Omitting the derivations, we give the expression for the stress-concentration coefficient at the edge of the cavity:

$$\varkappa = \pm{}^3/2\left[1 \pm \frac{9}{16}\frac{K}{3K + 4G}\frac{|p|}{G} + \frac{K(27K + 288Kg_3 + 360G + 384Gg_3)}{384(3K + 4G)^2}\frac{p^2}{G^2}\right].$$

Here the upper sign corresponds to tension, and the lower sign to compression at infinity. The magnitude of the stress concentration depends not only on the magnitude of the load, but also on its sign, as well as on the material constants.

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Note: Figure translations are in progress. See original paper for figures.