STATIONARY DIFFUSION ON "GRAY" BODIES, IN PARTICULAR, ON A "GRAY" SPHERE

PHYSICS

1970

SovietRxiv

Abstract

Full Text

UDC 535.2+533.15+541.182.2/3

PHYSICS

V. I. SMIRNOV

STATIONARY DIFFUSION ON "GRAY" BODIES, IN PARTICULAR, ON A "GRAY"SPHERE

(Presented by Academician M. A. Leontovich on 24 VII 1969)

Stationary diffusion of particles in the vicinity of "gray" absorbing bodies (absorption coefficient less than 1) is considered. It is assumed that the transport equations and the boundary conditions are linear with respect to the distribution function. Such problems arise in the theory of transport of radiation, molecules, and colloidal particles.

Suppose that the problem for "black" bodies of the same configuration as the "gray" ones has been solved, i.e., that the distribution function $f_0(\mathbf{x}, \mathbf{v})$ is known, where $\mathbf{x}$ are the coordinates and $\mathbf{v}$ is the particle velocity. One can write f_0 in the form

$$f_0(\mathbf{x}, \mathbf{v}) = \iint E(\mathbf{x}, \mathbf{x}', \mathbf{v}, \mathbf{v}') R_0(\mathbf{x}', \mathbf{v}')\, d\mathbf{x}' d\mathbf{v}', \tag{1}$$

where R_0 is the primary flux, and E is an auxiliary function*. Radiation of the bodies that is independent of f_0 is included in R_0. It is assumed that the medium neither absorbs nor emits particles.

The basic idea of the approach to the problem of "gray" bodies is as follows: mentally label all particles by an index k according to the number of collisions with the "gray" bodies that did not lead to absorption**. After the first collision, a secondary source arises

$$R_1(\mathbf{x}, \mathbf{v}) = \iint F(\mathbf{x}, \mathbf{x}', \mathbf{v}, \mathbf{v}') f_0(\mathbf{x}', \mathbf{v}')\, d\mathbf{x}' d\mathbf{v}', \tag{2}$$

where F is a function characterizing the "gray" bodies and the medium. From particles with $k = 1$, particles with $k = 2$ are obtained analogously, and so on. As a result we obtain

$$f(\mathbf{x}, \mathbf{v}) = \sum_{k=0}^{\infty} f_k(\mathbf{x}, \mathbf{v}), \tag{3}$$

where the distribution function of order k is equal to

$$f_k(\mathbf{x},\mathbf{v}) = \iint G(\mathbf{x},\mathbf{x}'',\mathbf{v},\mathbf{v}'') f_{k-1}(\mathbf{x}'',\mathbf{v}'')\, d\mathbf{x}'' d\mathbf{v}'' \quad (k \geqslant 1),$$

$$G(\mathbf{x},\mathbf{x}'',\mathbf{v},\mathbf{v}'') = \iint E(\mathbf{x},\mathbf{x}',\mathbf{v},\mathbf{v}') F(\mathbf{x}',\mathbf{x}'',\mathbf{v}',\mathbf{v}'')\, d\mathbf{x}' d\mathbf{v}'. \tag{4}$$

Formulas (3) and (4) show that, in order to solve problems on "gray" bodies, it is sufficient to find the solution of the problem on "black" bodies of the same geometry and to determine the single auxiliary function G.

It is clear from (4) that as $k \to \infty$, $f_k \to c_k f_\infty(\mathbf{x},\mathbf{v})$, where c_k are constants, while f_∞ is determined only by the function G (f_∞ is normalized so that the flux created by the distribution f_∞ onto the "black" bodies is equal to the flux created by f_0)***.

---

* Here and below, integrals over the entire space in which the particles move, and over all particle velocities, are meant.

** This idea will probably also be useful in further generalizations: nonstationary problems, cases where the medium absorbs and emits, etc.

*** From the geometrical point of view, special cases are possible (specular reflection in vacuum) in which f_∞ will be periodic in k, or in general will have no limit, one or several, as $k \to \infty$. These cases are not realized in practice because of the imperfection of reflectors and the unattainability of an absolute vacuum.

Let us consider the simplest example. Particles diffuse inside a sphere of radius r, and are assumed to move only along the radius and to have velocity either v_0 or $-v_0$. Let the particle source

$$R_0 = v_0 \delta(\zeta - r)\delta(v + v_0), \tag{5}$$

which may be regarded as surface radiation, create the initial distribution (the solution of the "black sphere" problem)

$$f_0(\zeta, v) = a(\zeta, r)\delta(v - v_0) + b(\zeta, r)\delta(v + v_0). \tag{6}$$

From the stationarity condition (the sum of the primary and absorbed fluxes is equal to 0), $a(r,r) = 1$. Since the positive direction of the velocity is taken to be away from the center, $b(r,r) = 0$.

Considering the first collision of particles with the wall, we find

$$E(\zeta,\zeta',v,v') = \frac{1}{v_0}\left[a(\zeta,\zeta')\delta(v+v') + b(\zeta,\zeta')\delta(v-v')\right], \qquad (7)$$

$$F(\zeta',\zeta'',v',v'') = v_0(1-\alpha)\delta(\zeta''-r)\delta(\zeta''-\zeta')\delta(v'+v''),$$

where α is the absorption coefficient. According to (4),

$$G(\zeta,\zeta'',v,v'') = (1-\alpha)\delta(\zeta''-r)\left[a(\zeta,\zeta'')\delta(v-v'') + b(\zeta,\zeta'')\delta(v+v'')\right]. \qquad (8)$$

In finding f_k from (4), a decreasing geometric progression is obtained. Finally,

$$f(\zeta,v) = \frac{1}{\alpha} f_0(\zeta,v). \qquad (9)$$

Let us note that if diffusion occurs inside an arbitrary closed domain bounded by a surface with a constant absorption coefficient α, then, except for the special cases mentioned,

$$f(\mathbf{x},\mathbf{v}) \to \frac{c}{\alpha} f_\infty(\mathbf{x},\mathbf{v}), \qquad (10)$$

where c is a constant, and f_∞ was defined above. In the example considered, f_0 and f_∞ coincide because of the identity of the primary and secondary sources, and $c = 1$.

In solving some problems of diffusion to "gray" bodies, one may use a rather simple approximate method based on approximating all f_k ($k \geqslant 1$) by functions of the form $c_k f_\infty$, where c_k are constants. Let us consider diffusion to a "gray" sphere from an infinite homogeneous medium. This problem is important for the theory of radiation and neutron transport, as well as for the theory of condensational and coagulative growth of colloidal particles.

The problem of the "black" sphere (often called the Milne problem for a sphere) has been solved in neutron transport theory ([1–3]). The distribution function $f_0(\zeta,\mu)$ satisfies the equation

$$\mu\frac{\partial f_0}{\partial \zeta} + \frac{1-\mu^2}{\zeta}\frac{\partial f_0}{\partial \mu} = -\frac{f_0}{l} + \frac{1}{2l}\int_{-1}^{1} f_0(\zeta,\mu')\,d\mu', \qquad (11)$$

where $\mu = \cos\theta$ (θ is the angle between the velocity and the radius vector), and l is the mean free path of the particles. The neutron velocity v between collisions with atoms of the medium (which does not enter the stationary equation (11)) is taken to be constant, and the scattering of neutrons by atoms is isotropic. The boundary conditions are: $f_0(\infty,\mu) = \text{const}$; $f_0(r,\mu) = 0$, $0 \leq \mu \leq 1$. At large distances from the sphere, (11) is equivalent to the diffusion equation. In the diffusion approximation the concentration satisfies the equation $\Delta n_d = 0$ with boundary conditions

$$n_d(\infty) = n_\infty = \text{const}; \qquad \lambda l \, dn_d/d\zeta|_{\zeta=r} = n_d(r), \tag{12}$$

where λ is the so-called linearly extrapolated length, depending on $l/r = \text{Kn}$ (the Knudsen number in molecular physics). The total flux onto

for a "black" sphere is equal to

$$I_0 = -4\pi D r n_\infty/(1 + \lambda \text{Kn}), \tag{13}$$

where $D = vl/3$ is the diffusion coefficient.

The asymptotics of $\lambda(\text{Kn})$ were found by Davison ([2]):

$$\begin{aligned} &\text{Kn} \ll 1: \quad \lambda = 0.7104 + 0.5047\text{Kn} + 0.2336\text{Kn}^2 + \cdots, \\ &\text{Kn} \gg 1: \quad \lambda = {}^4/_3 - {}^5/_9\text{Kn}^{-1} + 0.9782\text{Kn}^{-2}\ln\text{Kn} + \cdots. \end{aligned} \tag{14}$$

(the subsequent terms are omitted here). Sahni determined λ for intermediate Kn ([3]).

Let us find the particle concentration and the linearly extrapolated length $\lambda(a)$ for a "gray" sphere emitting particles*. Denote by j_k, i_k, and r_k the intensities of the fluxes of particles incident on the sphere, absorbed, and reflected, respectively, after k collisions with the sphere without absorption. Obviously,

$$i_k = a_k j_k, \qquad r_{k+1} = -(1 - a_k) j_k, \qquad j_l = a_l r_l,$$

$$n_l = r_l \varphi_l(\zeta) \qquad (k \geq 0, \; l = k + 1), \tag{15}$$

where a_k is the mean absorption coefficient, defined by the relation

$$a_k j_k = \int \alpha(\mathbf{v})(\mathbf{vn}) f_k(\mathbf{v}) \, d\mathbf{v} \tag{16}$$

($\mathbf{n}$ is the unit outward normal to the sphere), and φ_l is a certain function of the coordinates. As k and l increase, all a_k, a_l, and φ_l tend to limiting values (see above on the isomorphism of f_k as $k \to \infty$).

As a first approximation, assume that the distribution function of the particles reflected at the sphere is isotropic for $0 < \mu < 1$ and equal to 0 for the remaining μ; then a_k, a_l, and φ_l do not depend on k and l:

$$a_k = a, \qquad a_l = a, \qquad \varphi_l(\zeta) = \varphi(\zeta). \tag{17}$$

To determine the error introduced by this assumption, it is necessary to have additional information about $\alpha(\mathbf{v})$ and $f_k(\mathbf{v})$ in each separate case.

The intensities of the absorbed fluxes form a decreasing geometric progression, so that

$$i = \sum_{k=0}^{\infty} i_k = \frac{j_0 a}{1-(1-a)a}, \qquad j_0 = \frac{I_0}{4\pi r^2}. \tag{18}$$

In the limiting case $a \to 0$, the velocity distribution will be isotropic everywhere, the concentration constant, and then direct calculation of i gives

$$i = -{}^1/{}_4 v a n_\infty. \tag{19}$$

Comparison with (18) makes it possible to find a, and thereby also the particle flux onto a "gray" non-emitting sphere, which is determined from (13) with λ replaced by $\lambda(a)$ ([5])**

$$\lambda(a) = \lambda + 4(1-a)/3a. \tag{20}$$

From the condition that, as $a \to 0$, $n(\zeta) \to n_\infty$, we find the concentration $n(\zeta)$ (a special case of the more general formula for an emitting sphere is given below). If the sphere emits particles, then at a certain equilibrium concentra-

* It is assumed that the "grayness" is due to reflection from the surface of the sphere, and not to its being "shot through" straight across. The latter case was considered by D. F. Zaretskii and D. D. Odintsov ([4]).

** Formula (20) was obtained earlier by other methods by Amouyal, Benoist, and Horowitz ([6]), and independently (in another, but equivalent, form) by G. I. Marchuk ([7]), as the author learned after completion of this work.

of concentration n_s at an infinite distance from the sphere, dynamic equilibrium sets in, when the concentration is constant and the flux is zero. Hence it is easy to obtain the flux onto a "gray" emitting sphere and the concentration in the form

$$I = -4\pi D r(n_\infty - n_s)/[1 + \lambda(\alpha)\mathrm{Kn}], \tag{21}$$

$$\frac{n(\xi) - n_s}{n_\infty - n_s} = 1 - \left(1 - \frac{n_0(\xi)}{n_\infty}\right)\left(\frac{1+\lambda \mathrm{Kn}}{1+\lambda(\alpha)\mathrm{Kn}}\right)^*. \tag{22}$$

The concentration in the vicinity of a "black" sphere was determined by Sahni [3] for $\mathrm{Kn} = 0.5;\ 1;\ 2$ and 5, and by Le Caine [8] for a plane boundary ($\mathrm{Kn} = 0$). From geometrical considerations, for $\xi \ll l,\ \mathrm{Kn} \to \infty$,

$$n_0(\xi) \xrightarrow[\mathrm{Kn}\to\infty]{} \frac{1}{2} n_\infty \left(1 - \sqrt{1 - r^2/\xi^2}\right). \tag{23}$$

The flux intensity onto the sphere in the limiting cases is equal to:

$$\mathrm{Kn} \ll 1: \quad I \simeq -\frac{4\pi D r (n_\infty - n_s)}{1 + (4/3\alpha - 0.6229)\mathrm{Kn}}, \tag{24}$$

$$\mathrm{Kn} \gg 1: \quad I \simeq \frac{\pi r^2 v (n_\infty - n_s)}{1 + \frac{1}{3}\alpha\,\mathrm{Kn}^{-1}}. \tag{25}$$

Formulas (24) and (25) coincide with the approximate Fuchs formula known in aerosol theory [9,10] in the zeroth approximation in Kn and Kn^{-1}, respectively, but differ in higher approximations. The difference in the flux intensities onto the sphere reaches approximately 8% at $\mathrm{Kn} \simeq 2$.

From (22), at $\xi = r$ one obtains the relative jump of concentration at the surface of the sphere, expressed for the first time otherwise than through the concentration gradient. In the limiting cases we have:

$$\frac{\mathrm{Kn}}{\alpha} \ll 1: \quad \Delta = \frac{n(r) - n_s}{n_\infty - n_s} = \left[\frac{1}{\sqrt{3}} + \frac{4(1-\alpha)}{3\alpha}\right]\mathrm{Kn} + o\left(\frac{\mathrm{Kn}}{\alpha}\right), \tag{26}$$

$$\mathrm{Kn} \gg 1: \quad \Delta = 1 - \alpha/2 + o(\mathrm{Kn}^{-1}). \tag{27}$$

The concentration in the diffusion approximation is equal to

$$n_d = n_s + (n_\infty - n_s)(1 - r_d/\xi); \qquad r_d = r/[1 + \lambda(\alpha)\mathrm{Kn}]. \tag{28}$$

For $\mathrm{Kn} \to 0$ and $\alpha \to 1$, $n_d(r) \to 1.23 n_0(r)$, while for $\mathrm{Kn} \to \infty$, $n_d(r) = 2n_0(r)$. With the same error the concentration jump is determined in the diffusion approximation. As α decreases, the error decreases.

The problem of convective diffusion to a "gray" sphere can be solved analogously.

The author thanks N. A. Fuchs and A. D. Galanin for useful discussions and advice.

Central Aerological Observatory
Dolgoprudnyi, Moscow oblast

Received
12 VII 1969

References

[1] R. E. Marshak, Phys. Rev., **71**, 443 (1947).
[2] B. Davison, Proc. Phys. Soc. A, **64**, 881 (1951).
[3] D. C. Sahni, J. Nucl. Energy, **20**, A/B, No. 11/12, 915 (1966).
[4] D. F. Zaretskii, D. D. Odintsov, in: *Reactor Construction and Reactor Theory*, 1955, p. 279.
[5] B. I. Smirnov, *Trudy Tsentral' n. aerologich. observ.*, issue 55, 86 (1964).
[6] A. Amouyal, P. Benoist, J. Horowitz, J. Nucl. Energy, **6**, No. 1/2, 79 (1957).
[7] G. I. Marchuk, *Numerical Methods for Calculating Nuclear Reactors*, Moscow, 1958.
[8] J. Le Caine, Canad. J. Res. A, **28**, No. 3, 242 (1950).
[9] N. A. Fuchs, ZhETF, **4**, issue 7, 747 (1934).
[10] N. A. Fuchs, *Evaporation and Growth of Drops in a Gaseous Medium*, Publishing House of the Academy of Sciences of the USSR, 1958.

$*$ The distribution function $f(\xi,\mu)$ in approximation (17) also satisfies (22), with the replacements $n(\xi) \to f(\xi,\mu)$, $n_0(\xi) \to f_0(\xi,\mu)$, $n_\infty \to f_\infty = n_\infty/2$, $n_s \to f_s = n_s/2$.

Note: Figure translations are in progress. See original paper for figures.