VIBROSTABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.92

MATHEMATICS

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(Presented by Academician I. G. Petrovskii on 3 IV 1970)

1. Consider some element N, whose state is described by a vector quantity x. We shall assume that the total effect of all external actions can be determined by a single scalar quantity u. In the simplest cases the state x is some function of the quantity u (for example, the stress x in an elastic element is a function of the elongation u). In more complicated cases the quantity u does not uniquely determine the state x (for example, the stress x in a plastic element is not determined by its elongation u). Describing the dependence of x on u in these cases often causes difficulties: one has to speak of multivalued functions, deal with hysteresis phenomena, etc. In situations where deterministic processes are considered, one can avoid the indicated difficulties if the state x is regarded as a function $x(t)$ of time t, determined by the entire law $u(t)$ of variation of the quantity u. Below we call the function $u(t)$ a control. It is clear that the operator W, which determines the state $x(t) = Wu(t)$ from the control $u(t)$, is Volterra. Recall that an operator W is called Volterra if, for any two controls $u_1(t)$ and $u_2(t)$ $(0 \le t \le T)$, from $u_1(t) = u_2(t)$ for $0 \le t \le t_0$ it follows that the functions $Wu_1(t)$ and $Wu_2(t)$ also take identical values on the interval $0 \le t \le t_0$.

A broad class of examples of the situation described above is given by scalar or vector differential equations

$$dx/dt = f[t, x; u(t), u'(t)], \tag{1}$$

whose solutions $x(t)$ (for each fixed initial value) are uniquely determined by the control $u(t)$. Of course, instead of (1) one could consider more complicated equations, for example equations with deviating arguments. We restrict ourselves to equations (1) with a right-hand side containing the control $u(t)$ and its derivative $u'(t)$, because it is precisely to such equations that the analysis of many classical mechanical and physical systems with hysteretic nonlinearities leads (systems with plastic elements, with clearances, etc.).

The Volterra operator W, which determines the state $x(t)$ from the control $u(t)$, is usually specified initially only on sufficiently narrow classes of functions—most often on piecewise-monotone and piecewise-smooth continuous controls. Then the problem arises of extending it continuously to the space $C[0,T]$ of functions continuous on some interval $[0,T]$. If such an extension is possible, we shall say that the operator W is vibrostable. In the case when W is determined by equation (1), we shall say that the differential equation (1) is vibrostable.

In other words, equation (1) is vibrostable if, for every smooth control $u(t)$, the solution of equation (1) is uniquely determined by the initial value and if, from the uniform convergence of a sequence of controls $u_n(t)$ to some control $u_*(t)$ (which may

it no longer has any smoothness properties) it follows that the solutions $x_n(t) = Wu_n(t)$ converge uniformly to some function $x_*(t)$. The function $x_*(t) = \widetilde{W}u_*(t)$ may then be regarded as a generalized solution of equation (1) under the control $u_*(t)$.

Theorem 1. *Let equation (1) be vibro-stable. Let the function $f(t,x;u,v)$ be continuous in the totality of its variables.*

Then $f(t,x;u,v)$ admits the representation

$$f(t,x;u,v) = \varphi(t,x,u)v + \psi(t,x,u). \tag{2}$$

If the operator W is vibro-stable, then its extension to the space of continuous functions will be called a **hysterant operator**. An element N, whose change of states is described by a hysterant, will be called a **hysteron**. We note that the term "hysterant operator" was previously used by B. M. Daryinskii, I. V. Emelin, P. P. Zabreiko, E. M. Lifshits, and the authors in a more special case—when studying the Prager–Ishlinskii model of a plastic body.

2. From Theorem 1 it follows that equation (1) with a continuous right-hand side describing a hysteron necessarily has the form

$$dx/dt = \varphi[t,x,u(t)]u'(t) + \psi[t,x,u(t)]. \tag{3}$$

Theorem 2. *Let the function $\varphi(t,x,u)$ and its derivatives $\varphi'_t(t,x,u)$ and $\varphi'_x(t,x,u)$ satisfy a Lipschitz condition with respect to the variable x. Let the function $\psi(t,x,u)$ satisfy a Lipschitz condition with respect to the variables x and u. Let all the indicated functions be continuous in the totality of their variables, and let $\varphi'_x(t,x,u)$ be bounded.*

Then the differential equation (3) is vibro-stable. The hysterant operator determined by this equation satisfies a Lipschitz condition.

The conditions of Theorem 2 ensure, in particular, uniqueness and nonlocal extendability of solutions of equation (3) under various controls. This part of the restrictions is easy to weaken, but in doing so one has to complicate substantially the formulation of the theorem. We do not do this, since for applications what matters (as is clear from the considerations of the following paragraphs) are the values of the functions $\varphi(t, x, u)$ and $\psi(t, x, u)$ only for values of the variables from bounded domains.

For the actual construction of the values of the hysterant operator under the conditions of Theorem 2, one may proceed, for example, as follows. First one must determine the solutions $x = Q(u, x_0, u_0, t)$ of the problem

$$dx/du = \varphi(t, x, u), \qquad x(u_0) = x_0,$$

in which time t is regarded as a parameter. Then the solution $x(t)$ of equation (3) must be sought in the form

$$x(t) = Q[u(t), z(t), u(0), t].$$

To determine the function $z(t)$, one must solve the equation

$$dz/dt = Q'_t\{u(0), Q[u(t), z, u(0), t], u(t), t\}+$$
$$+ Q'_x\{u(0), Q[u(t), z, u(0), t], u(t), t\}\psi\{t, Q[u(t), z, u(0), t], u(t)\}$$

with the initial condition $z(0) = x_0$.

3. The mathematical description of phenomenological models of systems of the type of elastic-plastic elements leads to the problem of vibro-stability of equations (1) with discontinuous right-hand sides. In this case the discontinuities, as a rule, have a special character.

Usually the discontinuity surfaces separate a domain Ω, at whose interior points the change of the state $x(t)$ is determined by a differential-

equation with continuous right-hand side. If equation (1) with a discontinuous right-hand side is vibro-stable, then this right-hand side in the domain Ω has the form (2). The behavior on the discontinuity surfaces is described in a more complicated way.

In this section we restrict ourselves to considering the scalar equation (1), whose right-hand side has two discontinuity surfaces. Although in applications only solutions whose trajectories lie between the discontinuity surfaces are used, it is more convenient to study them by considering equation (1) for all values of the variables.

Let two continuously differentiable functions $l_-(t, u)$ and $l_+(t, u)$ be given, satisfying the inequality

$$l_-(t, u) < l_+(t, u). \tag{4}$$

Set

$$m_-(t,u,v) = \frac{\partial}{\partial t} l_-(t,u) + v \frac{\partial}{\partial u} l_-(t,u),$$

$$m_+(t,u,v) = \frac{\partial}{\partial t} l_+(t,u) + v \frac{\partial}{\partial u} l_+(t,u).$$

Consider the differential equation (1) with right-hand side

$$f_1(t,x,u,v) = \begin{cases} \max\{f_0(t,x,u,v),\, m_-(t,u,v)\}, & \text{for } x \le l_-(t,u), \\ f_0(t,x,u,v), & \text{for } l_-(t,u) < x < l_+(t,u), \\ \min\{f_0(t,x,u,v),\, m_+(t,u,v)\}, & \text{for } x \ge l_+(t,u), \end{cases} \tag{5}$$

where $f_0(t,x,u,v)$ is the right-hand side of some equation (2) satisfying the conditions of Theorem 2.

Since the function (5) is discontinuous, it becomes necessary to define the concept of a solution. We shall define a solution as an absolutely continuous function whose right derivative exists at every point and is equal to the right-hand side of the equation. With this definition of a solution, equation (1) with right-hand side (5), for any smooth control, has, for every initial value $x(0) = x_0$, a unique (in the direction of increasing t) and nonlocally continuable solution. Each solution coincides with the unique solution of the integral equation

$$x(t) = x_0 + \int_0^t f_1[s, x(s), u(s), u'(s)]\, ds.$$

We also note that the solution $x(t)$ satisfies the inequalities

$$l_-[t,u(t)] \le x(t) \le l_+[t,u(t)]$$

for all $t > 0$, if these inequalities hold for $t = 0$.

Theorem 3. *The differential equation (1) with right-hand side (5) is vibrostable. The hysterant determined by this equation satisfies the Lipschitz condition.*

The hysterant constructed from equation (1) with right-hand side (5) depends on the functions $l_-(t,u)$ and $l_+(t,u)$; in connection with this it is convenient to use the notation $W[l_-, l_+]$.

Theorem 4. *The operator-hysterant $W[l_-, l_+]$, defined by equation (1) with right-hand side (5), satisfies the local Lipschitz condition with respect to the variables l_- and l_+.*

The last theorem makes it possible, by means of a limiting passage, to generalize the definition of the hysteresis operator $W[l_-, l_+]$ to the case of arbitrary continuous functions $l_-(t, u)$ and $l_+(t, u)$ satisfying condition (4).

The authors express their gratitude to E. A. Lifshits for discussion of the results.

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Received
26 III 1970

Note: Figure translations are in progress. See original paper for figures.