ON A COMPLETE SYSTEM OF SOLUTIONS IN THE THEORY OF SHALLOW SHELLS

THEORY OF ELASTICITY

1970

SovietRxiv

Abstract

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UDC 539.4

THEORY OF ELASTICITY

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ON A COMPLETE SYSTEM OF SOLUTIONS IN THE THEORY OF SHALLOW SHELLS

The solution of boundary-value problems in the theory of shells is considerably complicated by the absence of complete systems of solutions of the corresponding equations. The only exceptions are circular cylindrical and spherical shells, as well as very shallow shells with a rectangular plan, for which particular solutions are known respectively in semigeodesic, polar, and Cartesian coordinates ([1]).

In the present paper, on the basis of integral representations of solutions of the equations of the theory of shallow shells, a certain system of regular solutions is constructed, complete with respect to any finite simply connected domain on the surface of the shell.

1. The general solution of the system of differential equations of the technical theory of shallow shells can be represented in the form

$$F(z,\zeta)=\sum_{i=0}^{1}\left\{a_{iG}i(z-z_0,\zeta-\zeta_0)+\int_{z_0}^{z}G_i(z-t,\zeta-\zeta_0)\varphi_i(t)\,dt+\right.$$
$$\left.+\int_{\zeta_0}^{\zeta}G_i(z-z_0,\zeta-\tau)\psi_i(\tau)\,d\tau\right\}, \tag{1.1}$$

where $\varphi_i(z)$ and $\psi_i(\zeta)$ $(i=0,1)$ are arbitrary analytic functions of their arguments; a_i are arbitrary constants, and the kernels $G_i(z,\zeta)$ are determined by the formulas

$$G_0(z,\zeta)=G_0(\zeta,z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}\,g_k'(\zeta),$$

$$G_1(z,\zeta)=G_1(\zeta,z)=\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}\,g_k(\zeta). \tag{1.2}$$

The functions $g_k(\zeta)$ can be represented in various forms:

a) in the form of series

$$g_k(\zeta) = \sum_{s=0}^{\infty} \frac{\zeta^{s+1}}{(s+1)!} C_{k,s}(\delta), \tag{1.3}$$

where

$$C_{2k,2s} = a_{k,s} = (k+s)! \sum_{j=0}^{\min(k,s)} \frac{(2\delta)^{2j}}{(2j)!(k-j)!(s-j)!}, \qquad C_{2k,2s+1} = C_{2k+1,2s} = 0,$$

$$C_{2k+1,2s+1} = b_{k,s} = (k+s+1)! \sum_{j=0}^{\min(k,s)} \frac{(2\delta)^{2j+1}}{(2j+1)!(k-j)!(s-j)!};$$

b) in the form of products

$$g_k(\zeta) = e^{\zeta} P_k(\zeta) + (-1)^{k+1} e^{-\zeta} P_k(-\zeta), \tag{1.4}$$

where $P_k(\zeta)$ are known polynomials of degree k.

The stress function $U(x,y)$ and deflections $w(x,y)$ are expressed in terms of the solution $F(z,\zeta)$ as follows:

$$F(z,\zeta) = F_1(z,\zeta) + iF_2(z,\zeta), \tag{1.5}$$

where

$$U(x,y) = F_1(z,\zeta), \qquad w(x,y) = \varepsilon^* F_2(z,\zeta), \qquad \varepsilon^* = \sqrt{\frac{12(1-\mu^2)}{Eh^2}}, \qquad |\alpha| \leqslant 1,$$

$$z = \frac{\beta\sqrt{i}}{a}(x+iy), \qquad \zeta = \frac{\beta\sqrt{i}}{a}(x-iy), \qquad \beta = \frac{\sqrt{\varepsilon(1-\alpha)}}{4},$$

$$\varepsilon = \frac{a^2}{Rh}\sqrt{12(1-\mu^2)}, \qquad \alpha = \frac{R}{R_1};$$

E, μ, h are the Young' s modulus, Poisson' s ratio, and shell thickness; R, R_1 are the radii of curvature of the middle surface; x, y are Cartesian coordinates; a is a characteristic linear dimension.

Thus, the stress function and deflections in the shell, and consequently all forces and displacements, are expressed in terms of the complex function $F(z,\zeta)$ given by the representation (1.1).

2. Introduce the functions

$$\Phi(z,\zeta) = L^0_{z,\zeta}\{\mu_0(z-z_0)\} =$$

$$= \frac{a_0}{2} G(z-z_0,\zeta-\zeta_0) + \int_{z_0}^{z} G(z-t,\zeta-\zeta_0)\mu_0(t-z_0)\,dt,$$

$$\psi(z,\zeta) = L^0_{z,\zeta}\{\mu_1(z_0-z)\} =$$

$$= \frac{a_1}{2} G(z_0-z,\zeta_0-\zeta) + \int_{z}^{z_0} G(t-z,\zeta_0-\zeta)\mu_1(z_0-t)\,dt, \tag{2.1}$$

$$\Phi^*(z,\zeta) = L^0_{\zeta,z}\{\nu_0(\zeta-\zeta_0)\} =$$

$$= \frac{a_0}{2} G(z-z_0,\zeta-\zeta_0) + \int_{\zeta_0}^{\zeta} G(z-z_0,\zeta-\tau)\nu_0(\tau-\zeta_0)\,d\tau,$$

$$\psi^*(z,\zeta) = L^0_{\zeta,z}\{\nu_1(\zeta_0-\zeta)\} =$$

$$= \frac{a_1}{2} G(z_0-z,\zeta_0-\zeta) + \int_{\zeta}^{\zeta_0} G(z_0-z,\tau-\zeta)\nu_1(\zeta_0-\tau)\,d\tau,$$

where the kernel has the form

$$G(z,\zeta) = G_0(z,\zeta) + \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \zeta}\right) G_1(z,\zeta).$$

By virtue of (1.1), the functions Φ, Ψ, Φ^*, and Ψ^* are solutions.

The kernel in the representations (2.1) can be written in the form

$$G(z,\zeta) = e^{z+\zeta} \sum_{k=0}^{\infty} \frac{z^k}{k!} \omega_k(\zeta) = G(\zeta, z), \tag{2.2}$$

where $\omega_k(\zeta)$ is a known function.

For example, if $\delta = 1$ (a circular cylindrical shell), then

$$\omega_k(\zeta) = \zeta^k / k!. \tag{2.3}$$

Construct the following solutions of the equations of the theory of shallow shells:

$$\Phi_\gamma(z,\zeta) = L^0_{z,\zeta}\left\{\frac{z^{\gamma-1}}{\Gamma(\gamma)}e^z\right\};\qquad \Psi_\gamma(z,\zeta) = L'_{z,\zeta}\left\{\frac{(-z)^\gamma}{\Gamma(\gamma)}e^{-z}\right\},$$

$$\Phi^*_\gamma(z,\zeta) = L^0_{\zeta,z}\left\{\frac{\zeta^{\gamma-1}}{\Gamma(\gamma)}e^\zeta\right\},\qquad \Psi^*_\gamma(z,\zeta) = L'_{\zeta,z}\left\{\frac{(-\zeta)^{\gamma-1}}{\Gamma(\gamma)}e^{-\zeta}\right\},\tag{2.4}$$

where for the time being we shall assume that $\operatorname{Re}\gamma > 0$.

Implementing the operators (2.4), we find, taking into account (2.1) and (2.2),

$$\Phi^*_\gamma(z,\zeta) = \Phi_\gamma(\zeta,z),\qquad \Psi_\gamma(z,\zeta) = \Phi_\gamma(-z,-\zeta),$$

$$\Psi^*_\gamma(z,\zeta) = \Phi_\gamma(-\zeta,-z),\qquad \Phi_\gamma(z,\zeta) = e^{z+\zeta}\sum_{k=0}^{\infty}\frac{z^{k+\gamma}\omega_k(\zeta)}{\Gamma(k+\gamma+1)},\tag{2.5}$$

$\Gamma(n)$ is Euler' s gamma function.

Thus, if the function $\Phi(z,\zeta)$ is a solution, then the functions $\Phi(\zeta,z)$, $\Phi(-z,-\zeta)$, and $\Phi(-\zeta,-z)$ will also be solutions.

Formulas (2.5) give the analytic continuation of the integrals in (2.4) to all values of the parameter γ, and therefore the restriction on γ can be removed.

For $\gamma = -n$ $(n = 1,2,...)$, we readily obtain from (2.5)

$$\Phi_{-n}(z,\zeta) = e^{z+\zeta}\sum_{k=0}^{\infty}\frac{z^k}{k!}\,\omega_{k+n}(\zeta).\tag{2.6}$$

The solutions (2.5) may be interpreted as generalized positive powers, since for $\gamma = n$ $(n = 0,1,...)$ the functions (2.5) have, at the point $z = 0$ $(\zeta = 0)$, a zero of multiplicity n. For $\gamma = 0$, the generalized powers coincide with the corresponding kernels.

It is clear that the system of powers Φ_n, Φ^*_n, Ψ_n, and Ψ^*_n $(n = 0,1,...)$ constitutes a system of regular solutions that is complete with respect to any finite simply connected domain.

As an example, let us consider the case of a circular cylindrical shell $(\delta = 1)$. From (2.5), taking into account (2.3), we find

$$\Phi_\gamma(z,\zeta) = \left(\frac{z}{\zeta}\right)^{\gamma/2} e^{z+\zeta}I_\gamma(2\sqrt{z\zeta}),\qquad \Psi_\gamma(z,\zeta) = \left(\frac{z}{\zeta}\right)^{\gamma/2} e^{-z-\zeta}I_\gamma(2\sqrt{z\zeta}),$$

$$\Phi^*_\gamma(z,\zeta) = \left(\frac{\zeta}{z}\right)^{\gamma/2} e^{z+\zeta} I_\gamma(2\sqrt{z\zeta}), \qquad \Psi^*_\gamma(z,\zeta) = \left(\frac{\zeta}{z}\right)^{\gamma/2} e^{-z-\zeta} I_\gamma(2\sqrt{z\zeta}), \tag{2.7}$$

where $I_\gamma(t)$ is the modified Bessel function of order γ. The representation of the solutions in the form (2.7) is more symmetric than the usual form for representing solutions for a cylindrical shell.

In an analogous way one can construct negative generalized powers, i.e., solutions having pole-type singularities at a prescribed point.

Solutions of type (2.5) may be used in solving boundary-value problems for a shallow shell in the form of a dome resting on a circular plan.

Received14 X 1969

CITED LITERATURE

1. E. I. Grigolyuk, L. A. Fil' shtinskii, *Perforated Plates and Shells and Problems Associated with Them. A Survey of Results*, TsAGI, Moscow, 1967.

Note: Figure translations are in progress. See original paper for figures.