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V. I. ZUBOV

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Abstract

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V. I. ZUBOV

ON A PROBLEM OF OPTIMAL STABILIZATION

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In the present note a problem of optimal stabilization is considered in the form of a differential game, and conditions for existence and a method for constructing optimal controls in analytic form are given.

Let us first consider the linear system of differential equations

$$\dot{X} = PX + QU \tag{1}$$

and the functional

$$I(X_0, U) = \int_0^\infty W^{(2)}\, dt, \tag{2}$$

where

$$W^{(2)} = X^*AX + X^*BU + U^*B^*X + U^*CU.$$

We shall assume that the elements of the matrices A, B, C, C^{-1}, P, and Q are real, continuous, and bounded functions of time, given for $t > 0$.

Put $X = (X_1, X_2)$ and $U = (U_1, U_2)$. We shall assume that the vectors X_i, U_i have dimensions n_i and r_i, respectively. In accordance with this the matrix C may be represented in the form

$$C = \begin{pmatrix} C_1 & D \\ D^* & C_2 \end{pmatrix}.$$

We shall make an assumption that is essential for what follows. Namely, we shall suppose that the quadratic form $U_1^*C_1U_1$ is positive definite, while $U_2^*C_2U_2$ is negative definite.

Definition 1. A control $U(t, X)$ is called **admissible** if: 1) $U(t, X) = M(t) \cdot X$, where $M(t)$ is a matrix with real, continuous, and bounded coefficients, given for $t > 0$; 2) system (1) under the control $U = M(t) \cdot X$ has a uniformly asymptotically stable equilibrium position $X = 0$ of exponential type.

Definition 2. An admissible control $U_0 = M_0(t) \cdot X$ is called **optimal** if

$$I(X_0, U_0) = \min_{U_1} \max_{U_2} I(X_0, U) = \max_{U_2} \min_{U_1} I(X_0, U)$$

for every initial vector X_0. Here $\min\max$ and $\max\min$ are taken over all such controls U_1 and U_2 that form an admissible control $U = (U_1, U_2)$.

Remark. The optimal control U_0 may thus be regarded as an equilibrium situation in a differential game of two persons X_1, X_2, whose sets of admissible strategies are given by Definition 1.

Theorem 1. In order that an optimal control U_0 exist, it is necessary and sufficient that there exist a real continuous-

a bounded matrix θ, defined for $t > 0$, satisfying the equation

$$\dot{\theta} + \theta Q C^{-1} Q^* \theta + \theta(P - QC^{-1}B^*) + (P - QC^{-1}B^*)^* \theta - A + BC^{-1}B^* = 0 \quad (3)$$

and such that the control $C^{-1}(Q^*\theta - B^*) \cdot X$ is admissible. In this case $U_0 = C^{-1}(Q^*\theta - B^*) \cdot X$.

Let us next consider the nonlinear system of differential equations

$$\dot{X} = PX + QU + \sum_{m=2}^{\infty} F^{(m)} = G(t, X, U), \quad (4)$$

and the functional

$$\int_0^{+\infty} \sum_{m=2}^{\infty} W^{(m)} = I(X_0, U). \quad (5)$$

Here $F^{(m)}$ and $W^{(m)}$ are homogeneous forms of degree m with respect to the components of the vectors X and U, with real, continuous, bounded coefficients, defined for $t > 0$. We shall henceforth assume that the series $W = \sum W^{(m)}$, $F = \sum F^{(m)}$ converge in some fixed neighborhood of the point $X = 0$, $U = 0$ uniformly with respect to $t > 0$.

Definition 3. A control $U(t, X)$ is called **admissible** if:

$$1) \quad U(t, X) = \sum_{m=1}^{\infty} U^{(m)};$$

2) $U^{(1)}(t, X)$ is an admissible control in the sense of Definition 1.

Here we assume that the series representing the admissible control converges uniformly with respect to $t \geq 0$ in some fixed neighborhood of the point $X = 0$.

Definition 4. An admissible control $U_0(t, X)$ is called **optimal** for system (4) with respect to the functional (5) if there exists some neighborhood of the point $X = 0$ such that for any initial value X_0 from this neighborhood one has

$$I(X_0, U_0) = \min_{U_1} \max_{U_2} I(X_0, U) = \max_{U_2} \min_{U_1} I(X_0, U).$$

Here, as above, $\min\max$ and $\max\min$ are computed for those U_1 and U_2 that provide an admissible control $U = (U_1, U_2)$ in the sense of Definition 3.

Theorem 2. If there exists a real continuous bounded solution θ of equation (3), defined for $t > 0$, such that the control $M_0 \cdot X$ is admissible in the sense of Definition 1,

$$M_0 = C^{-1}(Q^* \theta - B^*),$$

and the system $\dot{X} = P_0 X$, $P_0 = P + QM_0$, is proper, then there exists an optimal control

$$U_0(t, X) = \sum_{m=1}^{\infty} U_0^{(m)}$$

for system (4) with respect to the functional (5), representable in the form of series uniformly convergent for $t \geq 0$. Here, as above, $U_0^{(m)}$ are homogeneous forms of degree m in the components of the vector X.

Consider the system of equations

$$\frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial x} G(t, X, U) = -\frac{\partial G^*}{\partial x} \lambda + \frac{\partial W}{\partial x}, \tag{6}$$

$$\frac{\partial W}{\partial U} = \lambda^* \frac{\partial G}{\partial U}. \tag{7}$$

Here λ is a vector of dimension n; the partial derivatives with respect to the vector X and the vector U are defined in the natural way as matrices of the corresponding dimensions.

Theorem 3. *If the conditions of Theorem 2 are satisfied, then the system (6), (7) has a unique solution in the form of convergent series*

$$\lambda = \sum \lambda^{(m)}, \qquad U = \sum U^{(m)}, \tag{8}$$

$$\lambda^{(1)} = -2\theta X, \quad U^{(1)} = M_0 \cdot X,$$

uniformly with respect to $t \geq 0$*, and the series (8) gives the desired optimal control.*

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Note: Figure translations are in progress. See original paper for figures.