IDEMPOTENTS OF CROSSED PRODUCTS

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1970

SovietRxiv

Abstract

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UDC 519.48

MATHEMATICS

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IDEMPOTENTS OF CROSSED PRODUCTS

(Presented by Academician P. S. Novikov on 21 IV 1970)

Let G be an arbitrary group, and let K be an associative ring with identity. Suppose that there are given a single-valued mapping σ of the group G into the group of automorphisms of the ring K and a family $\rho = \{\rho_{g,h} \mid g, h \in G\}$ of invertible elements of the ring K, with the following relations satisfied:

$$\rho_{g_1,\, g_2 g_3} \rho_{g_2, g_3} = \rho_{g_1 g_2,\, g_3} \rho^{g_3 \sigma}_{g_1, g_2};$$

$$\alpha^{g_1 \sigma g_2 \sigma} = \rho^{-1}_{g_1, g_2} \alpha^{(g_1 g_2)\sigma} \rho_{g_1, g_2}$$

for all $\alpha \in K$ and $g_1, g_2, g_3 \in G$. The family ρ is called a system of factors.

Associate to each element $g \in G$ a symbol t_g, and consider the set W of all possible sums of the form

$$\sum_{g \in G} t_g \alpha_g \qquad (\alpha_g \in K),$$

in each of which only finitely many coefficients α_g are nonzero. The equality

$$\sum_{g \in G} t_g \alpha_g = \sum_{g \in G} t_g \beta_g$$

holds if and only if $\alpha_g = \beta_g$ for all $g \in G$. The set W becomes an associative ring if the operations of addition and multiplication are defined as follows:

$$\sum_{g \in G} t_g \alpha_g + \sum_{g \in G} t_g \beta_g = \sum_{g \in G} t_g (\alpha_g + \beta_g);$$

$$t_g t_h = t_{gh} \rho_{g,h};$$

$$\alpha t_g = t_g \alpha^{g\sigma} \qquad (\alpha \in K),$$

and, for arbitrary elements, multiplication is defined on the basis of the distributive law. This ring is called the crossed product of the group G and the ring K with system of factors ρ and mapping σ, and is denoted by (G, K, ρ, σ). A number of properties of this ring can be found in the papers ([1],[2]).

If σ maps the group G to the identity automorphism of the ring K, then the crossed product (G, K, ρ, σ) is called a crossed group ring, and we shall denote it by (G, K, ρ). Moreover, if the system of factors ρ is trivial, i.e. $\rho_{g,h} = 1$ for all $g, h \in G$, then the crossed product is a group ring, and we shall denote it by KG.

Let $A(K)$ be the group of automorphisms, and $B(K)$ the group of inner automorphisms of the ring K. The kernel of the mapping σ is called the kernel of the composite mapping

$$G \to A(K) \to A(K)/B(K).$$

If

$$x = \sum_{i=1}^{n} t_{g_i} a_i \qquad (a_i \neq 0, \quad i = 1, 2, \dots, n)$$

is an element of the ring (G, K, ρ, σ), then the subgroup L_x of the group G, generated by the elements $g_1, g_2, \dots, g_n$, is called the supporting subgroup of the element x.

This paper studies the structure of the supporting subgroups of idempotents of the crossed product (G, K, ρ, σ).

Theorem 1. For an arbitrary crossed product (G, K, ρ, σ) of a group G and a ring K, the supporting subgroup of every central idempotent of the ring (G, K, ρ, σ) is a finite normal subgroup of the group G.

The assertion on the finiteness of the supporting subgroups of central idempotents was put forward by Rudin and Schneider in ([3]), where, by methods of the theory of Banach algebras, the structure of the supporting subgroups of such idempotents in group algebras over the field of complex numbers is studied.

For some classes of rings K, Theorem 1 admits the following strengthening.

Theorem 2. A central idempotent of the crossed product (G, K, ρ, σ) of a group G and a commutative ring K without zero divisors (or K an arbitrary field) has a finite supporting subgroup belonging to the kernel of the mapping σ, and some prime divisor of the order of the supporting subgroup is invertible in the ring K, if $\operatorname{char} K = 0$.

If K is a ring of characteristic p, then from Theorem 2 it follows, by the Passman-Osima theorem (^{4}), that in the expression of every central idempotent of KG only those elements of the group G occur whose order is not divisible by p.

A ring K is called biregular if every principal two-sided ideal in the ring K is generated by a central idempotent.

If G is an arbitrary group, then it is known that the set of all elements of finite order having a finite number of conjugates in the group G is a normal subgroup of the group G, which we denote by G^+.

Theorem 3. Let KG be the group ring of an arbitrary group G over any biregular ring K, in which the order of every element $g \in G^+$ is invertible. The ring KG decomposes into a direct sum of indecomposable two-sided ideals if and only if the group G^+ is finite and the ring K satisfies the maximality condition for two-sided ideals.

Theorem 4. The group ring KG decomposes into a direct sum of minimal two-sided ideals if and only if

1. The group G is finite and its order is invertible in K.
2. The ring K decomposes into a direct sum of minimal two-sided ideals.

Theorem 5. The group ring of an arbitrary group G over a commutative ring K is biregular if and only if G is a locally normal group, K is biregular, and the order of every element of the group G is invertible in the ring K.

Theorem 5 remains valid if, instead of the commutativity condition on K, one requires the maximality condition for two-sided ideals.

We investigate the structure of crossed products containing idempotents only with finite supporting subgroups. Corollaries 1 and 2 generalize, respectively, Theorems 4.3 and 3.4 of Rudin and Schneider from (^{3}).

Let (G, K, ρ, σ) be an arbitrary crossed product of a group G and a ring K. Denote by $M(G, K)$ the set of all such elements g of finite order of the group G that

1. The order of the element g is invertible in K.
2. There exists an invertible element $\varepsilon \in K$ such that g and $t_g\varepsilon$ have the same orders.

If g has order n, then $(t_g\varepsilon)^n = t_1\rho_{1,1}^{-1}$ if and only if

$$\rho_g^{-1} = \varepsilon^{(g\sigma)n-1}\varepsilon^{(g\sigma)n-2}\dots\varepsilon^{g\sigma}\varepsilon \qquad \left(t_g^n = t_1\rho_{1,1}^{-1}\rho_g\right),$$

i.e. ρ_g is the norm of an element of K in the cyclic subring $(\langle g\rangle, K, \rho, \sigma)$.

Theorem 6. *Let all idempotents in the ring (G, K, ρ, σ) have finite support subgroups. Then the set $M(G, K)$ generates a locally finite normal subgroup of* *the group G. Moreover, if there exists an element $g \in M(G,K)$ that generates a noninvariant cyclic subgroup of the group G, then G is a locally finite group.*

Corollary 1. *Let G be a non-locally finite group and let the group ring KG have idempotents only with finite support subgroups, and let L be the subgroup generated by the set $M(G,K)$. Then*

1. *Every subgroup of the group L is normal in G.*
2. *If, for every n that is the order of an element of L, there exists in the ring K an invertible element of order n, then L belongs to the center of the group G.*
3. *If K is a field and L is a Hamiltonian group, then the algebra KL contains no nontrivial nilpotent elements.*

Theorem 7. *Let (G, K, ρ, σ) be the crossed product of a right-orderable group G and a commutative Noetherian ring K. If every primary ideal of the ring K is σ-admissible, then every idempotent of the ring (G, K, ρ, σ) has a trivial support subgroup.*

Theorem 8. *Let the center of the group G contain a finitely generated subgroup H such that G/H is a right-orderable group, and let K be a commutative Noetherian ring. If $\rho_{g,h} = \rho_{h,g}$ for any two permutable elements g and h of the group G, then all idempotents of the ring (G, K, ρ) have finite support subgroups.*

Corollary 2. *Let the center of the group G contain a finitely generated subgroup H such that G/H is a right-orderable group, and let K be a commutative ring. Then every idempotent of the group ring KG has a finite support subgroup.*

The problem of the existence of only trivial idempotents in group rings was first studied by S. D. Berman. He proved that the integral group ring ZG of a finite group G contains only trivial idempotents. It follows immediately from this that the integral group ring of a group approximable by finite groups has this property. Therefore it is natural to suppose that if the group ring KG of an arbitrary group G over a commutative integral domain has a nontrivial idempotent, then the order of some element of the group G is invertible in K. The validity of this supposition is confirmed by the result of Coleman (5) and by the following assertion:

Theorem 9. *The group ring KG of a locally nilpotent group G over a commutative integral domain K contains nontrivial idempotents if and only if there exists in the group G an element g of prime order p such that the element p is invertible in the ring K.*

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Received
26 III 1970

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Note: Figure translations are in progress. See original paper for figures.