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Abstract

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ON ONE INTEGRAL OF THE EQUATION OF INDUCTION OF A MAGNETIC FIELD FOR A FLOW OF DISSIPATIVE PLASMA

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Between the magnetic field and the field of the mass-averaged velocity of a plasma flow, under certain conditions, there exists a definite relation. This relation is the simplest integral of the differential equation of induction, based on general properties of the equations of induction and momentum transfer.

The induction equation for a plasma with constant magnetic viscosity ν_m has the form (^{1})

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot}(\mathbf{v} \times \mathbf{B}) - \nu_m \operatorname{rot}\operatorname{rot}\mathbf{B}.$$

A simple transformation, taking into account Maxwell' s equation

$$\operatorname{div}\mathbf{B} = 0,$$

and the continuity equation for an incompressible plasma

$$\operatorname{div}\mathbf{v} = 0,$$

allows it to be represented in the form

$$\partial \mathbf{B}/\partial t + (\mathbf{v}\nabla)\mathbf{B} = (\mathbf{B}\nabla)\mathbf{v} + \nu_m \nabla^2 \mathbf{B},$$

or, in projections on the axes,

$$\frac{\partial B_i}{\partial t} + u_k \frac{\partial B_i}{\partial x_k} = B_k \frac{\partial}{\partial x_k} u_i + \nu_m \frac{\partial}{\partial x_k}\left(\frac{\partial B_i}{\partial x_k} + \frac{\partial B_k}{\partial x_i}\right). \qquad (1)$$

The equation of momentum transfer has the form

$$\rho\left(\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}\right)=-\frac{\partial p}{\partial x_i}+\varepsilon_{ikll}j_{kB}l-\frac{\partial \pi_{ik}}{\partial x_k}, \tag{2}$$

where

$$\pi_{ik}=-\eta\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right) \tag{3}$$

are the elements of the viscous-stress tensor;

$$\varepsilon_{ikll}j_{kB}l=\partial T_{ik}/\partial x_k \tag{4}$$

are the projections of the electric body force;

$$T_{ik}=\frac{1}{\mu}B_{iB}k-\frac{B^2}{2\mu}\delta_{ik} \tag{5}$$

are the elements of the Maxwell stress tensor.

Substituting (3)–(5) into (2), we represent the momentum-transfer equation in the form

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=-\frac{1}{\rho}\frac{\partial p^0}{\partial x_i}+\nu\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)+\frac{1}{\mu\rho}B_k\frac{\partial B_i}{\partial x_k}, \tag{6}$$

where $p^0=p+B^2/2\mu$ is the total pressure, equal to the sum of the hydrodynamic and magnetic pressures.

For $\partial p^0/\partial x_i=0,\ \nu=\nu_m$, the induction equation is satisfied by the relation

$$B_i=\sqrt{\mu\rho}\,u_i+C_i, \tag{7}$$

where C_i is the integration constant for the relation connecting the projections **B** and **v** onto the axis x_i.

As is known, a unique connection between the magnetic field and the velocity field occurs in the case of a frozen-in field of plasma with infinite conductivity. The physical meaning of the integral found is that, under the indicated conditions (the magnetic Prandtl number ν/ν_m equal to unity, absence of a total-pressure gradient), the electric currents arising in the plasma create magnetic fields which, like a frozen-in field, are determined only by the plasma velocity, despite the presence of dissipative processes due to viscosity and finite conductivity.

It is interesting to note the great generality of the obtained integral of the induction equation, which holds both for steady and for unsteady plane and spatial plasma flows with arbitrary orientation of the magnetic field.

Let us consider the application of (7) to the boundary-layer problem, when the axis $x_i = x$ is parallel to the wall and $\partial p^0/\partial x = 0$. Then

$$B_x = \sqrt{\mu\rho}\, u + C.$$

Taking at the inner boundary of the boundary layer the conditions

$$B_x = B_{xw}, \qquad u = 0, \qquad y = 0,$$

we obtain

$$B_x = \sqrt{\mu\rho}\, u + B_{xw}.$$

Applying this relation to the outer boundary of the boundary layer, where

$$B_x = B_{xs}, \qquad u = U_s, \qquad y = \infty,$$

we find

$$B_{xs} = \sqrt{\mu\rho}\, U_s + B_{xw}.$$

Consequently,

$$(B_x - B_{xw})/(B_{xs} - B_{xw}) = u/U_s. \tag{8}$$

Thus, in the present case the profiles of the jumps of the tangential component of the magnetic field and the velocity profiles are similar.

The existence of profile similarity often makes it possible to establish a connection between physical quantities characteristic of the process under consideration.

Thus, from the similarity of the velocity profiles and the temperature-jump profiles there follows the well-known relation between friction and heat transfer, called the Reynolds analogy and forming the basis of the so-called hydrodynamic theory of heat transfer.

It is easy to see that from (8) there follows a relation between the shear stress and the electric current in the plasma. Indeed, differentiating (8) with respect to y, we obtain

$$\frac{\eta}{B_{xs} - B_{xw}} \frac{\partial B_x}{\partial y} = \frac{\eta}{U_s} \frac{\partial u}{\partial y},$$

where η is the coefficient of viscosity.

In the boundary-layer approximation

$$\partial v/\partial x < \partial u/\partial y, \qquad \partial B_y/\partial x < \partial B_x/\partial y,$$

the friction stress and current density

$$\tau_{xy} = \eta \frac{\partial u}{\partial y}, \qquad j_z = -\frac{1}{\mu} \frac{\partial B_x}{\partial y},$$

so that

$$-\frac{\eta\mu}{B_{xs} - B_{xw}} j_z = \frac{\tau_{xy}}{U_s}.$$

Using

$$\nu = \nu_m = (\sigma\mu)^{-1}, \qquad \eta = \rho\nu$$

and denoting

$$j_z^0 = \frac{j_z}{\sigma U_s (B_{xs} - B_{xw})}, \qquad C_f = \frac{\tau_{xy}}{\rho U_s^2/2},$$

we obtain

$$j_z^0 = -{}^1/_2 C_f.$$

Relation (7) can be used in the study of the plasma boundary layer, plasma jets, ionized wakes, magnetospheric flows, etc.

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REFERENCES

[1] L. E. Kalikhman, *Elements of Magnetohydrodynamics*, Moscow, 1964.

Note: Figure translations are in progress. See original paper for figures.