ON THE THEORY OF NONSTATIONARY AUTOMATIC CONTROL SYSTEMS

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Abstract

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CYBERNETICS AND CONTROL THEORY

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ON THE THEORY OF NONSTATIONARY AUTOMATIC CONTROL SYSTEMS

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1. The perturbed motion of a nonstationary automatic control system, under slow variation of the parameters of the plant and of the control system, in the linear approximation can be represented by the equations

$$A(\tau)\,dx/dt = B(\tau)x + \sum_{\nu} a_\nu(\tau)\delta_\nu \qquad (|\det A| \geq c > 0),$$

$$\delta_\nu = \int_{-\infty}^{t} g_\nu(t-t',\tau')\,v_\nu(t',\varepsilon)\,dt', \qquad v_\nu = b_\nu(\tau)x(t,\varepsilon), \tag{1.1}$$

where $\tau = \varepsilon t$ is "slow time" ; x is a column matrix composed of perturbations of the motion parameters of the controlled plant; δ_ν is the displacement of the actuating element of the regulator; v_ν and g_ν are, respectively, the input signal and the impulse transient function of the regulator; A, B, a_ν are matrices of dynamic coefficients of type, respectively, $n\times n,\ n\times n,\ n\times 1$. We shall assume that A, B, a_ν, b_ν, and g_ν are differentiable with respect to their arguments as many times as necessary.

In this note a method is indicated for constructing solutions of system (1.1) in the form of asymptotic series, and approximate criteria are formulated for the so-called "local stability" of the unperturbed motion (the trivial solution of system (1.1)).

As applied to control systems in which the influence of the regulator on the controlled process is small, so that the solutions of the equations of the closed system are close to the solutions of the equations corresponding to immovable control elements ($\delta_\nu \equiv 0$), such a problem was considered in ([1]).

2. Let us introduce the matrix

$$U(\lambda,\tau) \equiv A^{-1}(\tau)B(\tau) + A^{-1}\sum_{\nu} a_\nu(\tau)b_\nu(\tau)R^{(\nu)}(\lambda,\tau),$$

where

$$R^{(\nu)}(\lambda, t) = \int_0^\infty g_\nu(s,t)e^{-\lambda s}\, ds$$

is the transfer function of the regulator with parameters frozen at the given instant of time t, and the determining equation is

$$\Delta(\lambda, \tau) \equiv |U(\lambda, \tau) - \lambda E_n| = 0 \tag{2.1}$$

(E_n is the identity matrix of order n).

Each root $\lambda_\sigma(\tau)$ of equation (2.1) is at the same time an eigenvalue of the matrix $U^{(\sigma)}(\tau) \equiv U(\lambda_\sigma, \tau)$, so that, if $\mu_j^{(\sigma)}(\tau)$ $(j = 1, \dots, n)$ are the eigenvalues of the matrix $U^{(\sigma)}$, then at least one of these functions coincides with $\lambda_\sigma(\tau)$. We shall restrict ourselves to consideration of the simplest case, when with the function $\lambda_\sigma(\tau)$ for any $\tau \in [0, L]$ there coincides one and the same isolated eigenvalue of the matrix $U^{(\sigma)}(\tau)$, for example $\mu_1^{(\sigma)}(\tau)$.

By K_σ and M_σ we shall denote, respectively, the column and row matrices defined by the equalities

$$U^{(\sigma)}(\tau)K_\sigma(\tau) = K_\sigma(\tau)\mu_1^{(\sigma)}(\tau), \qquad M_\sigma(\tau)U^{(\sigma)}(\tau) = M_\sigma(\tau)\mu_1^{(\sigma)}(\tau), \tag{2.2}$$

$$M_\sigma(\tau)K_\sigma(\tau) = 1.$$

We shall assume that as K_σ and M_σ one has taken those solutions of equations (2.2) which are differentiable as many times as the matrix $U^{(\sigma)}$ is differentiable. Denote

$$R_{ij}^{(\nu)}(\lambda, t) = \frac{\partial^{i+j} R^{(\nu)}(\lambda, t)}{\partial \lambda^i \partial t^j} = (-1)^i \int_0^\infty \frac{\partial^j g_\nu(s,t)}{\partial t^j} s^i e^{-\lambda s}\, ds,$$

$$S(\lambda, t) = A^{-1}(t) \sum_\nu a_\nu(t) b_\nu(t) R_{10}^{(\nu)}(\lambda, t). \tag{2.3}$$

Theorem. Let λ_σ be a root of the determining equation (2.1), and for all $\tau \in [0, L]$:
1) $\mu_1^{(\sigma)}(\tau) \equiv \lambda_\sigma(\tau)$, $\mu_j^{(\sigma)}(\tau) \neq \mu_1^{(\sigma)}(\tau)$ $(j = 2, \dots, n)$;
2) $M_\sigma S_\sigma K_\sigma \neq 1$ $(S_\sigma \equiv S(\lambda_\sigma, t))$.

Then the formal solution of system (1.1) corresponding to this root can be represented in the form

$$x_\sigma(t,\varepsilon) = \widetilde{K}_\sigma(\tau,\varepsilon) y_\sigma, \qquad dy_\sigma/dt = \tilde{\lambda}_\sigma(\tau,\varepsilon) y_\sigma, \tag{2.4}$$

where $\widetilde{K}_\sigma$ and $\tilde{\lambda}_\sigma$ are, respectively, a column matrix and a scalar function admitting the formal expansions

$$\widetilde{K}_\sigma(\tau,\varepsilon) = K_\sigma(\tau) + \sum_{k=1}^{\infty} \varepsilon^k K_\sigma^{[k]}(\tau), \qquad \tilde{\lambda}_\sigma(\tau,\varepsilon) = \lambda_\sigma(\tau) + \sum_{k=1}^{\infty} \varepsilon^k \lambda_\sigma^{[k]}(\tau). \tag{2.5}$$

Proof. Substitute into equations (1.1) the value of the vector x determined by equalities (2.4). We obtain

$$A(\tau)\left[\varepsilon\, d\widetilde{K}_\sigma(\tau,\varepsilon)/d\tau + \widetilde{K}_\sigma(\tau,\varepsilon)\tilde{\lambda}_\sigma(\tau,\varepsilon)\right] = B(\tau)\widetilde{K}_\sigma(\tau,\varepsilon) + \sum_\nu a_\nu(\tau) I_\sigma^{(\nu)}, \tag{2.6}$$

where

$$I_\sigma^{(\nu)} = \int_{-\infty}^{t} g_\nu(t-t',\tau') b_\nu(\tau') \widetilde{K}_\sigma(\tau',\varepsilon) \exp\left(\int_t^{t'} \tilde{\lambda}_\sigma \, dt''\right) dt'.$$

The integral $I_\sigma^{(\nu)}$ can be transformed to the form

$$\begin{aligned} I_\sigma^{(\nu)} &= b_\nu(\tau)\widetilde{K}_\sigma(\tau,\varepsilon) R_{00}^{(\nu)}(\tilde{\lambda}_\sigma,\tau) \\ &\quad + \varepsilon\Bigg\{\frac{d\left[b_\nu(\tau)\widetilde{K}_\sigma(\tau,\varepsilon)\right]}{d\tau} R_{10}^{(\nu)}(\tilde{\lambda}_\sigma,\tau) + b_\nu(\tau)\widetilde{K}_\sigma(\tau,\varepsilon) R_{11}^{(\nu)}(\tilde{\lambda}_\sigma,\tau) \\ &\qquad + \frac{1}{2} b_\nu(\tau)\widetilde{K}_\sigma(\tau,\varepsilon) \frac{d\tilde{\lambda}_\sigma(\tau,\varepsilon)}{d\tau} R_{20}^{(\nu)}(\tilde{\lambda}_\sigma,\tau)\Bigg\} + \varepsilon^2 \dots . \end{aligned} \tag{2.7}$$

The functions $R_{ij}^{(\nu)}(\tilde{\lambda}_\sigma,\tau)$, in turn, admit expansions

$$R_{ij}^{(\nu)}(\tilde{\lambda}_\sigma,\tau) = R_{ij}^{(\nu)}(\lambda_\sigma,\tau) + \sum_{k=1}^{\infty} \varepsilon^k \lambda_\sigma^{[k]} R_{i+1\,j}^{(\nu)}(\lambda_\sigma,\tau) +$$

$$+\frac{1}{2}\left(\sum_{k=1}^{\infty} \varepsilon^k \lambda_\sigma^{[k]}\right)^2 R_{i+2\,j}^{(\nu)}(\lambda_\sigma,\tau) + \dots \tag{2.8}$$

Taking into account relations (2.5), (2.7), and (2.8), we equate in equality (2.6) the coefficients of like powers of ε. We obtain

$$U^{(\sigma)}K_\sigma = K_\sigma \lambda_\sigma; \tag{2.9}$$

$$U^{(\sigma)}K_\sigma^{[k]} = K_\sigma^{[k]}\lambda_\sigma + (K_\sigma - SK_\sigma)\,\lambda_\sigma^{[k]} + D_\sigma^{[k-1]} \qquad (k = 1, 2, \dots). \tag{2.10}$$

Here $D_\sigma^{[k-1]}$ is a column matrix, known when $K_\sigma, \lambda_\sigma, \dots, \dots, K_\sigma^{[k-1]}, \lambda_\sigma^{[k-1]}$ are known. Thus, for example,

$$D_\sigma^{[0]} = \frac{dK_\sigma}{dt} - A^{-1}\sum_\nu a_\nu \left\{ \frac{d(b_\nu K_\sigma)}{dt} R_{10}^{(\nu)}(\lambda_\sigma, \tau) + +b_\nu K_\sigma \left[R_{11}^{(\nu)}(\lambda_\sigma, \tau) + \frac{1}{2}\frac{d\lambda_\sigma}{dt} R_{20}^{(\nu)}(\lambda_\sigma, \tau) \right] \right\}.$$

By virtue of (2.2) and condition 1) of the theorem, equality (2.9) is satisfied identically. We shall show that, with an appropriate choice of $K_\sigma^{[k]}$ and $\lambda_\sigma^{[k]}$, equalities (2.10) also become identities.

Let us first carry out some additional constructions. The square matrix $P_\sigma = K_\sigma M_\sigma$ is the projection matrix corresponding to the eigenvalue $\mu_1^{(\sigma)} = \lambda_\sigma$ of the matrix $U^{(\sigma)}$. By condition 1) of the theorem, the projection matrix corresponding to all the remaining eigenvalues of the matrix $U^{(\sigma)}$ is $P_{-\sigma} = E_n - P_\sigma$. The rank of the square matrix $P_{-\sigma}$ is $n-1$, and therefore it can be decomposed into factors $K_{-\sigma}$ and $M_{-\sigma}$ ($P_{-\sigma} = K_{-\sigma}M_{-\sigma}$)—matrices of types respectively $n \times n-1$ and $n-1 \times n$ of rank $n-1$, just like the matrix $P_{-\sigma}$, differentiable with respect to τ as many times as $U^{(\sigma)}$ is differentiable. The matrices $K_{-\sigma}$ and $M_{-\sigma}$ are related to each other and to the matrices K_σ, M_σ by the relations

$$M_{-\sigma}K_{-\sigma} = E_{n-1}, \qquad M_{-\sigma}K_\sigma = M_\sigma K_{-\sigma} = 0. \tag{2.11}$$

Further, if $K^{(\sigma)} \equiv (K_\sigma K_{-\sigma})$, $M^{(\sigma)} \equiv \binom{M_\sigma}{M_{-\sigma}}$, and

$$\Lambda^{(\sigma)} \equiv \begin{pmatrix} \lambda_\sigma & 0 \\ 0 & \Lambda_{-\sigma} \end{pmatrix},$$

where $\Lambda_{-\sigma} = M_{-\sigma}U^{(\sigma)}K_{-\sigma}$, then

$$U^{(\sigma)} = K^{(\sigma)}\Lambda^{(\sigma)}M^{(\sigma)}, \qquad M^{(\sigma)}K^{(\sigma)} = K^{(\sigma)}M^{(\sigma)} = E_n \tag{2.12}$$

(see ([2])). Note also that the eigenvalues of the matrix $\Lambda_{-\sigma}$ are the eigenvalues $\mu_j^{(\sigma)}$ $(j = 2, \dots, n)$ of the matrix $U^{(\sigma)}$.

Now multiply the k-th equality (2.10) on the left by $M^{(\sigma)}$, replacing $U^{(\sigma)}$ in it by expression (2.12). We obtain

$$\Lambda^{(\sigma)}Q_\sigma^{[k]} = Q_\sigma^{[k]}\lambda_\sigma + M^{(\sigma)}(K_\sigma - SK_\sigma)\lambda_\sigma^{[k]} + M^{(\sigma)}D_\sigma^{[k-1]}. \tag{2.13}$$

Here

$$Q_\sigma^{[k]} \equiv M^{(\sigma)}K_\sigma^{[k]} = \begin{bmatrix} M_\sigma K_\sigma^{[k]} \\ M_{-\sigma}K_\sigma^{[k]} \end{bmatrix} \equiv \begin{bmatrix} q_{\sigma\sigma}^{[k]} \\ Q_{-\sigma\,\sigma}^{[k]} \end{bmatrix}.$$

Since $\Lambda^{(\sigma)}$ is a quasi-diagonal matrix, equality (2.13) decomposes into the following two:

$$M_\sigma(K_\sigma - SK_\sigma)\lambda_\sigma^{[k]} + M_\sigma D_\sigma^{[k-1]} = 0,$$

$$\Lambda_{-\sigma}Q_{-\sigma\,\sigma}^{[k]} = Q_{-\sigma\,\sigma}^{[k]}\lambda_\sigma + M_{-\sigma}(K_\sigma - SK_\sigma)\lambda_\sigma^{[k]} + M_{-\sigma}D_\sigma^{[k-1]}. \tag{2.14}$$

By condition 2) of the theorem, the first equality (2.14) is solvable with respect to $\lambda_\sigma^{[k]}$:

$$\lambda_\sigma^{[k]} = -\pi_\sigma M_\sigma D_\sigma^{[k-1]} \qquad (\pi_\sigma = 1/(1 - M_\sigma S_\sigma K_\sigma)). \tag{2.15}$$

The matrix $\Lambda_{-\sigma}$ has no eigenvalues equal to λ_σ. Consequently, $\Lambda_{-\sigma} - \lambda_\sigma E_{n-1}$ is nonsingular, and from the second equality (2.14) one can determine the submatrix $Q_{-\sigma\,\sigma}^{[k]}$ of the matrix $Q_\sigma^{[k]}$:

$$Q_{-\sigma\,\sigma}^{[k]} = (\Lambda_{-\sigma} - \lambda_\sigma E_{n-1})^{-1}M_{-\sigma}(E_n + \pi_\sigma S_\sigma P_\sigma)D_\sigma^{[k-1]}. \tag{2.16}$$

As the other submatrix $q_{\sigma\sigma}^{[k]}$ of the matrix $Q_\sigma^{[k]}$, one may take an arbitrary scalar function differentiable the required number of times.

Knowing $Q_\sigma^{(k)}$, it is easy to determine the desired column matrix $K_\sigma^{(k)}$

$$K_\sigma^{[k]} = K^{(\sigma)}Q_\sigma^{[k]} = K_\sigma q_{\sigma\sigma}^{[k]} + K_{-\sigma}Q_{-\sigma\sigma}^{[k]}. \tag{2.17}$$

The recurrence relations (2.15)–(2.17) make it possible successively to determine the terms of the series (2.5), by means of which a particular solution (2.4) of system (1.1) is represented. The theorem is proved.

This theorem and the procedure described above for constructing the solution are readily generalized to the case where A and B are functions of τ and ε admitting, on $[0, L]$, expansions (convergent or, at least, asymptotic) in powers of ε.

3. If the conditions of the theorem are satisfied for each root λ_i $(i = 1, \dots, m)$ of the characteristic equation, then m particular solutions $x_1, x_2, \dots, x_m$ of the form (2.4) can be constructed. Among these solutions there may also be ones that coincide with one another (up to a constant factor), if the corresponding roots of the characteristic equation are identical. Let $x_1, \dots, x_l$ $(l \leq m)$ be all the distinct particular solutions. We shall assume that, in the course of their construction, the arbitrariness present in the choice of K_σ and $q^{(k)}_{\sigma\sigma}$ has been restricted by the condition

$$\widetilde{K}^*_\sigma K_\sigma = 1 \qquad (\sigma = 1, \dots, l). \tag{3.1}$$

Then, as follows from (2.4), the Euclidean norm of each column matrix x_σ will coincide with the modulus of the corresponding scalar function y_σ, and therefore

$$d\|x_\sigma\|/dt = \operatorname{Re}\, \tilde{\lambda}_\sigma \|x_\sigma\| \qquad (\sigma = 1, \dots, l). \tag{3.2}$$

If

$$\operatorname{Re}\, \tilde{\lambda}_\sigma(t_0) < 0. \qquad (\sigma = 1, \dots, l), \tag{3.3}$$

then, by continuity, Re $\tilde{\lambda}_\sigma(t) < 0$ $(\sigma = 1, \dots, l)$ within some finite interval $[t_0, t_0 + \Delta t]$, and therefore, according to (3.2), $\|x_\sigma(t)\| < \|x_\sigma(t_0)\|$ $(\sigma = 1, \dots, l,\ t \in [t_0, t_0 + \Delta t])$. Taking this into account, satisfaction of the inequalities (3.3) may be regarded as a sufficient condition for stability of the zero solution of system (1.1) on the finite interval $[t_0, t_0 + \Delta t]$ (local stability).

In accordance with (3.3), approximate stability criteria can be formulated by retaining in the expansions of the functions $\tilde{\lambda}_\sigma(t)$ a certain number of the first terms. The simplest criterion of this kind is obtained by retaining in the expansion of $\tilde{\lambda}_\sigma$ only the first term:

$$\operatorname{Re} \lambda_\sigma < 0 \qquad (\sigma = 1, \dots, l).$$

In this form, as is known, the stability criterion is obtained within the framework of the method of frozen coefficients.

Retaining in the expansions of $\tilde{\lambda}_\sigma$ the first two terms, we obtain a criterion that already takes into account the variability of the parameters of the plant and the controller:

$$\operatorname{Re}(\lambda_\sigma - \varepsilon \pi_\sigma M_\sigma D^{[0]}_\sigma) < 0 \qquad (\sigma = 1, \dots, l).$$

Further refinement can be obtained by adding subsequent terms of the expansions.

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Note: Figure translations are in progress. See original paper for figures.