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## Abstract

## Full Text

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MATHEMATICS

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# ON THE APPROXIMATE FINDING OF FIXED POINTS OF CONTINUOUS MAPPINGS

*(Presented by Academician L. V. Kantorovich, 16 VII 1969)*

The note discusses methods for constructing approximations to fixed points under the conditions of the theorems of Brouwer, Kakutani, and Lefschetz-Hopf ([1,4,5]), using the apparatus of simplicial subdivisions. They may find application in nonlinear analysis, as well as in the solution of certain problems of mathematical economics ([2–4]).

1. Let $T \subset E^n$ be a closed connected set and let $f : T \to T$ be a continuous mapping. The set of fixed points of $f$, obviously, coincides with $\varphi^{-1}(0)$, where $\varphi(x) = f(x) - x$. To describe the construction of passage to a discrete mapping–numbering–we consider $Z_{n+1} = \{1, 2, \ldots, n+1\}$ and a family of continuous vector-functions $v_i : T \to E^n,\ i \in Z_{n+1}$, such that for any $x \in T$ one has

$$\sum_{i=1}^{n+1} v_i(x) = 0, \qquad (v_i(x), v_j(x)) \leq 0$$

for $i \neq j$, and the vectors $\{v_i(x)\},\ 1 \leq i \leq n$, are linearly independent. Further, let

$$M_i = \{x \in T \mid (\varphi(x), v_i(x)) \leq 0\}, \qquad \mu(x) = \{i \in Z_{n+1} \mid x \in M_i\}.$$

**Lemma 1.**

$$\bigcap_{i=1}^{n+1} M_i = \varphi^{-1}(0), \qquad \bigcup_{i=1}^{n+1} M_i = T$$

and, consequently, $\mu(x) \neq \emptyset$ for all $x \in T$.

**Definition.** A mapping $\nu : T \to Z_{n+1}$ such that $(\forall x)\ \nu(x) \in \mu(x)$ is called a **numbering**. A set $D \subset T$ is called **representative** for $\mu$ if

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$$\bigcup_{x \in D} \mu(x) = Z_{n+1},$$

and a simplex $\sigma$ is representative if its set of vertices is so. A **normal** ([1,2]) simplex in the numbering $\nu$ is an $n$-dimensional simplex $\sigma^n$ (whose set of vertices is denoted by the same symbol) for which $\nu(\sigma^n) = Z_{n+1}$, and an $(n-1)$-dimensional simplex $\sigma^{n-1}$ is $(Cj)$-normal if

$$\nu(\sigma^{n-1}) = Z_{n+1} - \{j\}.$$

Obviously, a normal simplex is representative. Further, by Lemma 1 a one-point representative set is contained in $\varphi^{-1}(0)$. For the case when $T = (a_1, a_2, \dots, a_{n+1})$ is an $n$-dimensional simplex with vertices $a_i$, the classical numbering in the proof of Brouwer' s theorem using Sperner' s lemma is obtained with the aid of functions $v_i(x) = v_i^0$, where $v_i^0$ are the outer normals to the corresponding (not containing $a_i$) $(n-1)$-dimensional faces of the original simplex, with

$$\sum v_i^0 = 0, \qquad (v_i^0, v_j^0) \le 0.$$

This presupposes a special realization of the simplex $T$ in $E^n$. Below, in item 4, the inner normals of the corresponding faces are chosen as the $v_i^0$, and it then turns out that $\mu(x) \supset Z_{n+1} - I = CI$ for all $x \in \Gamma(I) = (a_i)_{i \in I}$, where $I$ is an arbitrary nonempty subset of $Z_{n+1}$, $\Gamma(I)$ is the face of the simplex $T$ determined by the vertices $a_i$ whose indices belong to $I$; in other words,

$$\Gamma(I) \subset \bigcap_{i \in CI} M_i.$$

2. The estimate given here for the closeness of a representative simplex to a fixed point is essentially based on Lebesgue' s lemma ([1,5]) on the existence, for a finite open covering of a metric compactum $K$, of a $\delta > 0$ (a Lebesgue number) such that every set $M \subset K$ with diameter less than $\delta$ is contained in one of the sets of the covering.

Let $x^*$ be an isolated fixed point for $f$, and let a closed neighborhood $U$ of this point contain no other fixed points. Put

$$P_\varepsilon = T \cap U \cap CS(x^*, \varepsilon), \quad Q = \{\varepsilon \mid S(x^*, \varepsilon) \subset U\},$$

$$\gamma(\varepsilon) = \min_{x \in P_\varepsilon} \max_{i \in Z_{n+1}} d(x, M_i), \quad \beta(p) = \sup\{\varepsilon \in Q \mid \gamma(\varepsilon) \le p\},$$

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where $S(x^*, \varepsilon)$ is the open sphere with center $x^*$ and radius $\varepsilon > 0$, the symbol $C$, as usual, denotes the complement of the corresponding set, and $d(x, M_i)$ is the distance from $x$ to $M_i$. Below, by the radius of a geometric simplex $\sigma^k = (a_1, a_2, \dots, a_{k+1})$ we mean

$$\rho = \min_{x \in \sigma} \max_{1 \le i \le k+1} d(x, a_i),$$

and the center of this simplex is considered to be a point $z$ such that

$$\max_i d(z, a_i) = \rho.$$

**Theorem 1.** *Let the closed set $U$ be a neighborhood of a fixed point $x^*$ of the mapping $f : T \to T$, and let $U$ contain no other fixed points of this mapping. Then, if $\sigma$ is a representative simplex of radius $\rho$, whose vertices and center $z$ lie in $T \cap U$, then $d(x^*, z) \le \beta(\rho)$.*

**Remark.** In a number of cases, for example if linearization of $f$ is admissible near $x^*$, the quantity $\gamma(\varepsilon)$ (the Lebesgue number of the open covering $\{CM_i\}$ of the compact set $P_\varepsilon$) and the function $\beta$ can be effectively estimated; in any case

$$\lim_{p \to 0} \beta(p) = 0.$$

**3.** The theorem of the preceding section may serve as a basis for constructing approximations to fixed points of a continuous self-mapping of a polyhedron by means of simplicial subdivisions. However, not every fixed point can be approximated in this way; this is not difficult to illustrate by examples for any dimension $n \ge 1$.

**Theorem 2.** *Let a fixed point $x^*$ of a continuous mapping $f$ of a polyhedron $T$ into itself have nonzero topological index, and let a polyhedral closed neighborhood of this point $U \subset T$ be given, with $U$ containing no other fixed points of the mapping $f$. Further, let $\delta$ be the Lebesgue number of the covering of the boundary of the set $U$ by the family of open sets $(CM_i)_{i \in Z_{n+1}}$. Then, for any subdivision of the polyhedron $U$, effected by a complex $K$ with diameter less than $\delta$, under any numbering $\nu$ constructed for the mapping $f$, there exists a normal simplex in the complex $K$.*

The proof uses the theorem on the index from paper ([6]).

**Corollary.** *Let the conditions of Theorem 2 be fulfilled, and let $K_m$ be a sequence of complexes with diameters $\delta_m < \delta$, $\delta_m \to 0$ as $m \to \infty$, $|K_m| = U$. Then, for any sequence $\{\sigma_m\}$ of representative simplices $\sigma_m \in K_m$, every sequence of points $x_m \in \sigma_m$ converges to the fixed point $x^*$ of the mapping $f$.*

As a further consequence, we obtain a statement concerning fixed points under the conditions of the Lefschetz-Hopf theorem. In doing so, it is necessary to take into account the presence of different variants of numberings of the points of the polyhedron, associated with its different realizations.

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**Theorem 3.** *Let $f : P \to P$ be a continuous mapping of a dimensionally homogeneous $n$-dimensional polyhedron, having only regular fixed points with respect to some complex $K$, and let the Lefschetz number $\Lambda_f \neq 0$. Then there exists a fixed point $x^* \in P$ to which the representative simplices converge (in the sense of the corollary to Theorem 2).*

The method of subdivisions can be justified for a broader class of problems, above all for the search for boundary or non-isolated fixed points and for point-to-set mappings.

**4.** We describe one algorithm for finding representative simplices. Let $T = (a_1, a_2, \dots, a_{n+1})$ be an $n$-dimensional simplex and let $K$ be its triangulation. Construct an abstract complex $K_1$, representing a refinement of the complex $K$. For this we introduce new vertices $\{b_i\}$, $i \in Z_{n+1}$, and for all

For $I \subset Z_{n+1}$, consider the collection of $n$-dimensional simplices $\tau(I)$, for each of which the vertices are $\{b_i\}$, $i \in CI$, and the vertices of some full-dimensional simplex $\sigma(I) \in K$ lying in the face $\Gamma(I) = (a_i)_{i \in I}$ of the original simplex $T$. All such simplices $\tau(I)$ and their faces are included in $K_1$. It can be shown that $K_1$ is a closed pseudomanifold. Let the mapping $\mu$, which assigns to the vertices $x$ of the complex $K$ subsets $\mu(x) \subset Z_{n+1}$, be such that, for all vertices $x$ from the subcomplex of the complex $K$ corresponding to the face $\Gamma(I)$, one has $\mu(x) \supset CI$. One way of constructing such a mapping $\mu$ is indicated at the end of Section 2. Extend the mapping $\mu$ to the set of vertices of $K_1$, setting for the new vertices $\mu(b_i) = \{i\}$, and consider some numbering $\nu$ of the vertices of the complex $K_1$, i.e., a single-valued function such that $\nu(x) \in \mu(x)$. The proposed algorithm is based on

**Lemma 2.** *If, for a nonempty $I \subset Z_{n+1}$, some simplex $\tau(I) \in K_1$ is normal in the numbering $\nu$, then the corresponding simplex $\sigma(I) \in K$ is representative for $\mu$.*

In the algorithm, as the initial simplex one takes the simplex $\tau(\emptyset) = (b_1, b_2, \dots, b_{n+1})$, which is, obviously, normal. Suppose now that there is some normal simplex $\sigma_0 \in K_1$. Fix $j \in Z_{n+1}$ and construct $n$-dimensional simplices $\sigma_q \in K_1$, each of which is obtained from the preceding one by deleting some vertex $w_{q-1}$ and including a vertex $w'_q$ distinct from it (uniquely determined), where $\nu(w_0) = j$, and, if $\nu(w'_q) \neq j$, then as $w_q$ one takes the unique vertex of $\sigma_q$ such that $\nu(w_q) = \nu(w'_q)$, $w_q \neq w'_q$. From the properties of pseudomanifolds and from the fact that a normal simplex contains one, and the others contain 0 or 2, $(Cj)$-normal faces, it follows that after a finite number of steps we obtain $\nu(w'_q) = j$. The simplex $\sigma_q$ corresponding to it and distinct from $\sigma_0$ is normal, and Lemma 2 is applicable to it.

The algorithm described has common features with the method proposed and successfully tested (for $n \sim 10$) in [7] for approximating fixed points under the conditions of Brouwer' s theorem. It can be shown that the family of primitive sets from [7] forms a simplicial complex $K$ such that the polyhedron $|K|$ is close to the original simplex $T$, and the algorithm in [7] is essentially one of the

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economical methods for finding a representative simplex in $K$. Therefore one cannot agree with the opposition of the algorithm from [7] to a method based on Sperner' s lemma. Moreover, from Theorem 2 given above one can obtain the sufficient condition, absent in [7], for the applicability of the method discussed there.

In conclusion, we note that the information report [8] mentions a recent talk by G. Kuhn devoted to the approximation of fixed points of a mapping of a simplex into itself by means of simplicial subdivisions.

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*Note: Figure translations are in progress. See original paper for figures.*

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