DISCONTINUOUS SOLUTIONS OF NONLINEAR MIXED PROBLEMS FOR ALMOST LINEAR HYPERBOLIC SYSTEMS IN THE PLANE

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Abstract

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DISCONTINUOUS SOLUTIONS OF NONLINEAR MIXED PROBLEMS FOR ALMOST LINEAR HYPERBOLIC SYSTEMS IN THE PLANE

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1°. In the rectangle $\bar{\Pi}_T = \{0 \le x \le 1,\ 0 \le t \le T\}$, $0 < T < \infty$, consider the mixed problem

$$\partial u/\partial t + \lambda\, \partial u/\partial x = P(x, t, u, q_1, \dots, q_n); \tag{1}$$

$$\partial q_i/\partial t - v_i\, \partial q_i/\partial x = Q_i(x, t, u, q_1, \dots, q_n); \tag{2}$$

$$u(x, 0) = u_0(x), \qquad q_i(x, 0) = q_{i,0}(x); \tag{3}$$

$$q_i(1, t) = 0; \tag{4}$$

$$H(t, u(0, t), q_1(0, t), \dots, q_n(0, t)) = 0, \tag{5}$$

where the constants $\lambda, v_i > 0,\ v_i \ne v_j$ for $i \ne j$, and the functions $P, Q_i \in C_2(\bar{\Pi}_T \times R^{n+1})$, $H \in C_2([0, T] \times R^{n+1})$, $u_0, q_{i,0} \in C_2[0, 1]$ satisfy the compatibility conditions necessary for the existence in $\bar{\Pi}_T$ of a solution of problem (1)–(5) of class C_1. For simplicity, suppose that $0 < T < (\lambda + \max v_i)^{-1}$.*

Following the ideas, developed in works ($^{1-4}$), on unique solvability in the small and continuation in t of solutions of mixed problems for quasilinear and almost linear hyperbolic systems in the plane, it is not difficult to establish the following proposition.

Theorem 1. Suppose that $H'_u(0, u_0(0), q_0(0)) \neq 0$. Then either A) in $\bar{\Pi}_T$ there exists a unique solution $(\dot{u}, \dot{q}) \in C_1(\bar{\Pi}_T)$ of problem (1)–(5), and

$$|H'_u(t, \dot{u}(0,t), \dot{q}(0,t))| > 0$$

for $0 \leq t \leq T$; or B) there is a $0 < T^* \leq T$ such that in $\bar{\Pi}_{T^*} = \{0 \leq x \leq 1,\ 0 \leq t < T^*\}$ there exists a unique solution $(\dot{u}, \dot{q}) \in C_1(\Pi_{T^*})$ of problem (1)–(5),

$$|H'_u(t, \dot{u}(0,t), \dot{q}(0,t))| > 0$$

for $0 \leq t < T^*$, and at least one of the following equalities holds:**

$$s \equiv \sup_{\Pi_{T^*}} |\dot{u}| + \sum_{i=1}^{n} \sup_{\Pi_{T^*}} |\dot{q}_i| = \infty, \qquad m \equiv \inf_{0 \leq t < T^*} |H'_u(t, \dot{u}(0,t), \dot{q}(0,t))| = 0.$$

Below we investigate the correctness of the formulation of problem (1)–(5) in $\bar{\Pi}_T$ under the assumption that assertion B) of Theorem 1 holds, but $T^* < T$, $s < \infty$, and, consequently, $m = 0$. In this situation, when inequalities (6), (7) are satisfied, problem (1)–(5) has in $\bar{\Pi}_{T^*+\Delta T}$, for arbitrarily small $\Delta T > 0$, not even a continuous generalized solution (c.g.s.), and increasing the smoothness and compatibility of the functions $P, Q_i, H, u_0, q_{i,0}$ does not lead to the existence of a c.g.s.

* The inequality $0 < T < (\lambda + \max v_i)^{-1}$ excludes the mutual influence of the boundary conditions (4) and (5) on the properties of the solution of problem (1)–(5) in $\bar{\Pi}_T$. Therefore all the results of this note hold under more general boundary conditions than (4).

** Examples show that all logical possibilities contained in assertion B) are realized.

In the present note, under the assumption that inequalities (6), (7) and condition b), formulated in 3°, are satisfied, a discontinuous solution (d.s.) of problem (1)–(5) in $\overline{\Pi}_\tau$ for $\tau > T^*$ is constructed, and theorems are given on the stability of the d.s. under small perturbations of the initial conditions and under perturbation of the function H by the term

$$\mu L(u(0,t), q(0,t)) \equiv \mu \left[\gamma_1 \partial u/\partial t + \gamma_2 \partial u/\partial x + \sum_{k=1}^{n} (\rho_1^k \partial q_k/\partial t + \rho_2^k \partial q_k/\partial x) \right]_{x=0},$$

where γ_i, ρ_i^k are constants, and μ is a small parameter.

Discontinuous solutions of nonlinear mixed problems for hyperbolic equations of second order were constructed in (5).

2°. Smoothness $(\dot{u}, \dot{q})$ in $\overline{\Pi}_{T^*}$. Absence of a d.s. in $\overline{\Pi}_\tau$ for $\tau > T^*$.

1. For $T^* < \tau \le T$, denote by

$$G^0_\tau = \{(x,t) \in \overline{\Pi}_\tau,\ 0 \le t \le \tau,\ 0 \le x \le \lambda t\}, \quad K^0_\tau = \overline{\Pi}_\tau \setminus G^0_\tau.$$

Let $\overline{\Pi}'_{T^*}$ be the rectangle $\overline{\Pi}_{T^*}$ with the "punctured point" $(0, T^*)$, and let $\mathcal{D} = \overline{\Pi}'_{T^*} \cap G^0_\tau$.

Below it is assumed throughout that

$$I \equiv \int_0^{T^*} \left|H'_u(\tau, \dot{u}(0,\tau), \dot{q}(0,\tau))\right|^{-1} d\tau < \infty. \tag{6}$$

Theorem 2. The functions $\dot{u} \in C(\overline{\Pi}_{T^*})$, $\dot{u} \in C_1(\overline{\Pi}'_{T^*})$, $\dot{q} \in C_1(\overline{\Pi}_{T^*})$,

$$\sup_D |\dot{u}_x(x,t)|\, \chi(t - x/\lambda) < \infty, \quad \sup_D |\dot{u}_t(x,t)|\, \chi(t - x/\lambda) < \infty,$$

where $\chi(t) = |H'_u(t, \dot{u}(0,t), \dot{q}(0,t))|$ for $0 \le t < T^*$.

From Theorem 2 and the equality $m = 0$ it follows that $H'_u(T^*, u^*, q^*) = 0$, where $u^* = \dot{u}(0, T^*)$, $q^* = \dot{q}(0, T^*)$. Below it is assumed throughout that

$$H''_{uu}(T^*, u^*, q^*) \ne 0. \tag{7}$$

2. Consider problem (1)–(5) in $\overline{\Pi}_\tau$ for $\tau > T^*$. Let $T^* < T_1 \le T$ be such that in $K^0_{T_1}$ there exists a unique solution $(\dot{u}, \dot{q}) \in C_1(K^0_{T_1})$ of problem (1)–(4).*

Definition 1. A vector-function $(u,q) \in C(\overline{\Pi}_\tau)$, $T^* < \tau \le T_1$, such that $u = \dot{u}$, $q = \dot{q}$ in K^0_τ, is called a d.s. of problem (1)–(5) in $\overline{\Pi}_\tau$ if $H(t, u(0,t), q(0,t)) = 0$ for $0 \le t \le \tau$, and at each point $(x,t) \in G^0_\tau$

$$u(x,t) = u(0, t - x/\lambda) + \int_{t-x/\lambda}^{t} P(\xi, \tau, u(\xi,\tau), q(\xi,\tau))\big|_{\xi = x(\tau,x,t)}\, d\tau, \tag{8}$$

$$q_i(x,t) = \dot{q}_i(\xi_i(x,t), \tau_i(x,t)) + \int_{\tau_i(x,t)}^{t} Q_i(\xi, \tau, u(\xi,\tau), q(\xi,\tau))\big|_{\xi = x_i^+(\tau,x,t)}\, d\tau, \tag{9}$$

where $x(\tau,x,t) = \lambda\tau + x - \lambda t$, $x_i^+(\tau,x,t) = -\nu_i\tau + x + \nu_i t$, and $(\xi_i(x,t), \tau_i(x,t))$ is the point of intersection of the characteristics $\xi = x_i^+(\tau,x,t)$, $\xi = x(\tau,0,0)$.

Theorem 3. Whatever $T^* < \tau \le T_1$, there is no d.s. of problem (1)–(5) in $\overline{\Pi}_\tau$.

3°. Construction of a d.s. of problem (1)–(5) in $\overline{\Pi}_\tau$ for $\tau > T^*$.

1. For $T^* < \tau \le T_1$, denote by

$$K^1_\tau = \{(x,t) \in \overline{\Pi}_\tau,\ T^* \le t \le \tau,\ 0 \le x \le \lambda(t - T^*)\}, \quad \mathcal{L}_\tau = G^0_\tau \setminus K^1_\tau,$$

$$\mathcal{D}_\tau = \mathcal{L}_\tau \cup K^0_\tau.$$

* Considering problem (1)–(5) in K^0_T, it is not difficult to establish a theorem of existence and uniqueness of a continuously differentiable solution, analogous to Theorem 1.

Fix an arbitrary $T^* < T_2 \le T_1$ such that in $\overline{\mathcal{L}}_{T_2}$ there exists a solution $(u,q) \in C(\overline{\mathcal{L}}_{T_2})$ of the system (8), (9), satisfying the equality $u(0,t) = \mathring{u}(0,t)$ for $0 \le t \le T^*$. By uniqueness in $\overline{\mathcal{L}}_{T_2}$ of the solution (u,q),

$$u = \mathring{u}, \quad q = \mathring{q} \quad \text{in } \overline{\mathcal{D}}.$$

In $\overline{\mathcal{D}}_{T_2}$ define the vector-function $(\tilde{u}, \tilde{q})$, setting $(\tilde{u}, \tilde{q}) = (u,q)$ in $\overline{\mathcal{L}}_{T_2}$, and $(\tilde{u}, \tilde{q}) = (\mathring{u}, \mathring{q})$ in $K^{\circ}_{T_2}$.

Theorem 4. The functions $\tilde{u} \in C(\overline{\mathcal{D}}_{T_2})$, $\tilde{u} \in C_1(\mathcal{D}_{T_2})$, $\tilde{q} \in C_1(\overline{\mathcal{D}}_{T_2})$,

$$\sup_{\mathcal{L}_{T_2}} |\tilde{u}_x(x,t)|\, \chi(t - x/\lambda) < \infty, \qquad \sup_{\mathcal{L}_{T_2}} |\tilde{u}_t(x,t)|\, \chi(t - x/\lambda) < \infty,$$

and the vector-function $(\tilde{u}, \tilde{q})$ satisfies equation (1) in $\mathcal{D}_{T_2}$, and equation (2) in $\overline{\mathcal{D}}_{T_2}$.

Below the vector-function $(\tilde{u}, \tilde{q})$ is denoted by $(\mathring{u}, \mathring{q})$.

2. Suppose

a) $H'_u(t, \mathring{u}(0,t), \mathring{q}(0,t)) > 0$ for $0 \le t < T^*$, $H''_{uu}(T^*, u^*, q^*) < 0$*; (10)

b) the equation $H(T^*, u, q^*) = 0$ has a real root $u^* < \overline{u}^* < \infty$ such that $H(T^*, u, q^*) \ne 0$ for $u^* < u < \overline{u}^*$, $H'_u(T^*, \overline{u}^*, q^*) > 0$.

Definition 2. A vector-function $(\overset{1}{u}, \overset{1}{q}) \in C_1(K^1_T)$, $T^* < T \le T_2$, such that

$$\overset{1}{u}(0, T^*) = \overline{u}^*, \qquad \overset{1}{q}(x(t,0,T^*),t) = \mathring{q}(x(t,0,T^*),t) \quad \text{for } T^* \le t \le T,$$

is called a solution of problem (1)–(5) in K^1_T, if it satisfies equations (1), (2) in K^1_T, and for $T^* \le t \le T$

$$H(t, \overset{1}{u}(0,t), \overset{1}{q}(0,t)) = 0, \qquad H'_u(t, \overset{1}{u}(0,t), \overset{1}{q}(0,t)) > 0.$$

Theorem 5. The following alternative holds:

A) in $K^1_{T_2}$ there exists a unique solution $(\overset{1}{u}, \overset{1}{q})$ of problem (1)–(5);

B) there is a $T^* < \widetilde{T} \le T_2$ such that, for any $\varepsilon > 0$, in $K^1_{\widetilde{T}-\varepsilon}$ there exists a unique solution $(\overset{1}{u}, \overset{1}{q})$ of problem (1)–(5), and at least one of the following relations holds:

$$\max_{K^1_{\widetilde{T}-\varepsilon}} |\overset{1}{u}| + \sum_{i=1}^{n} \max_{K^1_{\widetilde{T}-\varepsilon}} |\overset{1}{q}_i| \to \infty \quad \text{as } \varepsilon \to 0,$$

$$\inf_{T^* \le t < \widetilde{T}} H'_u(t, \overset{1}{u}(0,t), \overset{1}{q}(0,t)) = 0.$$

Set $T^0 = T_2$ in case A), and $T^0 = T^* + \theta(\widetilde{T} - T^*)$ in case B) ($0 < \theta < 1$ is an arbitrary fixed number).

Definition 3. The vector-function $(u^p, q^p) = (\mathring{u}, \mathring{q})$ in $\mathcal{D}_{T^0}$, $(u^p, q^p) = (\overset{1}{u}, \overset{1}{q})$ in $K^1_{T^0}$ is called an r.d. of problem (1)–(5) in $\overline{\Pi}_{T^0}$.

4. Stability of the r.d. under small perturbations of the function H. In $\overline{\Pi}_{T^0}$ consider the mixed problem (1′)–(5′), determined by equations (1)–(4) and the boundary condition

$$\mu L(u(0,t), q(0,t)) + H(t, u(0,t), q(0,t)) = 0, \quad \mu \ne 0, \tag{5'}$$

We note that $(\mathring{u}, \mathring{q})$ is a solution of problem (1′)–(5′) in $K^0_{T^0}$ for any μ.

Definition 4. A vector-function $(u, q) \in C_1(G^0_{T^0})$ such that

$$u(0,0) = u_0(0), \qquad q(x(t,0,0),t) = q(x(t,0,0),t) \quad \text{for } 0 \le t \le T^0$$

is called a solution of problem (1′)–(5′) in $G^0_{T^0}$, if it satisfies equations (1), (2) in $G^0_{T^0}$ and equation (5′) for $0 \le t \le T^0$.

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* The inequalities (10) do not restrict generality.

Set $\varepsilon = \mu(\gamma_1 - \lambda^{-1}\gamma_2)$. Let $\varepsilon > 0$ when $\mu \ne 0$.

Theorem 6. One can indicate an $\varepsilon^* > 0$ such that, for $0 < \varepsilon < \varepsilon^*$, in G_{T^0} there exists a unique solution $(u(\mu,x,t), q(\mu,x,t))$ of problem (1′)–(5′), and

$$\|u(\mu,\cdot)\|_{C(G_{T^0})} < C < \infty, \qquad \|q_i(\mu,\cdot)\|_{C_1(G_{T^0})} < C < \infty,$$

where C is a constant independent of μ.

Theorem 7. Whatever closed domain $\Omega \subset G_{T^0}$, having no common points with the characteristic $t = x/\lambda + T^*$, may be,

$$\|u(\mu,\cdot) - u^p(\cdot)\|_{C(\Omega)} + \|q_i(\mu,\cdot) - q_i^p(\cdot)\|_{C_1(\Omega)} \to 0 \quad \text{as } \mu \to 0.$$

If Ω has no common points with the characteristic $t = x/\lambda$, then

$$\|u(\mu,\cdot) - u^p(\cdot)\|_{C_1(\Omega)} \to 0 \quad \text{as } \mu \to 0.$$

5°. Stability of d.s. under small perturbations of the initial conditions. In $\overline{\Pi}_{T^0}$ consider problem (1)–(5) with initial conditions $u(x,0) = u_0(x)$, $q_i(x,0) = q_{i,0}(x)$, where the functions $u_0, q_{i,0} \in C_2[0,1]$ satisfy the compatibility conditions necessary for the existence in $\overline{\Pi}_{T^0}$ of a solution of problem (1)–(5) of class C_1. Denote by

$$\Delta = \|u_0 - \mathring{u}_0\|_{C_1[0,1]} + \sum_{i=1}^{n} \|q_{i,0} - \mathring{q}_{i,0}\|_{C_1[0,1]}.$$

Theorem 8. For every $\eta > 0$ there exists a $\delta(\eta) > 0$ such that if $\Delta < \delta(\eta)$, then one can indicate a τ^*, $|\tau^* - T^*| < \eta$, such that in $\Pi_{\tau^*} = \{0 \le x \le 1,\ 0 \le t < \tau^*\}$ there exists a unique solution $(\mathring{u}, \mathring{q}) \in C_1(\Pi_{\tau^*})$ of problem (1)–(5), $H'_u(t, \mathring{u}(0,t), \mathring{q}(0,t)) > 0$ for $0 \le t < \tau^*$, and the quantities $s < \infty$, $m = 0$, $I < \infty$ corresponding to the solution $(\mathring{u}, \mathring{q})$.

By Theorem 2 there exist

$$\overset{\vee}{u}{}^* = \lim_{t \to \tau^*} \mathring{u}(0,t), \qquad q^* = \lim_{t \to \tau^*} \mathring{q}(0,t).$$

Theorem 9. If Δ is sufficiently small, then $H''_{uu}(\tau^*, \overset{\vee}{u}{}^*, q^*) < 0$.

Theorem 10. If Δ is sufficiently small, then

1) the equation $H(\tau^*, u, q^*) = 0$ has a real root $\bar{u}^* < \bar{u}^* < \infty$ such that $H(\tau^*, \overset{\vee}{u}{}^*, q^*) \ne 0$ for $\overset{\vee}{u}{}^* < \overset{\vee}{u} < \bar{u}^*$, and $H'_u(\tau^*, \bar{u}^*, q^*) > 0$;

2) in $\overline{\Pi}_{T^0}$ there exists a unique d.s. (u^p, q^p) of problem (1)–(5)*.

Theorem 11. Whatever closed domain $\Omega \subset \overline{\Pi}_{T^0}$, having no common points with the characteristic $t = x/\lambda + T^*$, may be,

$$\|u^p - \mathfrak{u}^p\|_{C_1(\Omega)} + \|q_i^p - \mathfrak{q}_i^p\|_{C(\overline{\Pi}_T^0)} + \|q_i^p - \mathfrak{q}_i^p\|_{C_1(\Omega)} \to 0 \quad \text{as } \Delta \to 0.$$

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* We do not dwell on the obvious definition of a d.s. of problem (1)–(5) with initial conditions (u_0, q_0).

Note: Figure translations are in progress. See original paper for figures.

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