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MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 517.946.9

MATHEMATICS

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ON THE FIRST BOUNDARY-VALUE PROBLEM WITH WEIGHTED BOUNDARY CONDITIONS FOR DEGENERATE ELLIPTIC EQUATIONS OF HIGHER ORDERS

(Presented by Academician S. L. Sobolev on V. 6, 1970)

1°. Introduction. The question of studying boundary-value problems for elliptic equations in the class of unbounded solutions was posed by A. V. Bitsadze ([1]): it is required to find a solution regular in the domain which, with a certain weight tending to zero on approaching the boundary or a part of it, satisfies the prescribed boundary conditions. Boundary-value problems of this kind for second-order equations were investigated in ([2]-[4]) and others. In the present note the first boundary-value problem with weighted boundary conditions for degenerate elliptic equations of higher orders in a bounded domain is studied.

2°. Basic notation and definitions. E_n is n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$; $E_n^+ = \{x : x_n > 0\}$; $E_{n-1} = \{x : x_n = 0\}$; $x' = (x_1, \dots, x_{n-1})$; Ω is a domain in E_n possessing the following property: $\Omega = E_n^+ \cap \Delta$, where Δ is such a bounded domain in E_n, whose boundary Σ is a closed $(n-1)$-dimensional surface of smoothness order $m+1$, $m \geq 1$, and for which $\Delta \cap E_{n-1} \neq \emptyset$; $\Gamma_0 = \Delta \cap E_{n-1}$; $\Gamma_1 = \Sigma \cap E_n^+$; Γ_0' is the projection of Ω onto E_{n-1}; the multi-index $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ is used to denote differentiation only with respect to the variables of the set x': $D_{x'}^{\alpha}$.

In what follows we shall need the functional spaces V and $\dot{V}$.

Definition 1. By the **space** V we shall mean the totality of all functions u, defined in the domain Ω, for which there exist generalized (in the sense of S. L. Sobolev) derivatives of order m, and

$$p(u) = \left(\sum_{k=1}^{R} \sum_{|\alpha| \leq m_k} \left\| \mathfrak{L}_k^{(\alpha)} u \right\|^2_{L_{q_k}(\Omega)} \right)^{1/2} < \infty. \tag{1}$$

In the seminorm $p(u)$: 1) R, m_k are natural numbers; 2) $\max_{1\le k\le R} m_k = m$; 3) $1 < q_k < \infty$; 4) $\mathfrak{L}_k^{(\alpha)}$ is a differential operator of the form

$$\mathfrak{L}_k^{(\alpha)} = P_{k,0}^{(\alpha)} P_{k,1}^{(\alpha)} \cdots P_{k,m_k-|\alpha|-1}^{(\alpha)} P_{k,m_k-|\alpha|}^{(\alpha)},$$

where

$$P_{k,m_k-|\alpha|}^{(\alpha)} = x_n^{\gamma_{k,m_k-|\alpha|}{}^{(\alpha)}(x')} D_{x'}^{\alpha}, \qquad P_{k,j}^{(\alpha)} = x_n^{\gamma_{k,j}{}^{(\alpha)}(x')} \partial/\partial x_n$$

for $|\alpha| < m_k$, $0 \le j < m_k - |\alpha|$; 5) the functions $\gamma_{k,j}^{(\alpha)}(x')$ are defined on Γ_0', measurable and bounded.

Put

$$\mu_{k,j}^{(\alpha)}(x') = -j + \gamma_{k,0}^{(\alpha)}(x') + \cdots + \gamma_{k,j-1}^{(\alpha)}(x'), \qquad |\alpha| < m_k, \quad 1 \le j \le m_k - |\alpha|,$$

and, for simplicity of subsequent formulations, we shall assume that the following condition is fulfilled.

Condition 1. In the case when $|\alpha| < m_k - 1$, there exists an $\varepsilon > 0$ such that for any j_1 and j_2 satisfying the inequalities $j_1 \ne j_2$, $1 \le j_1 \le m_k - |\alpha|$, $1 \le j_2 \le m_k - |\alpha|$, the relation

$$\operatorname{mes}\left(\left\{x' \in \Gamma_0 : \left|\mu_{k,j_1}^{(\alpha)}(x') + q_k^{-1}\right| < \varepsilon\right\} \cap \left\{x' \in \Gamma_0 : \left|\mu_{k,j_2}^{(\alpha)}(x') + q_k^{-1}\right| < \varepsilon\right\}\right) = 0$$

holds.

Let

$$\Gamma_{k,j}^{(\alpha)} = \{x' \in \Gamma_0 : \mu_{k,j}^{(\alpha)}(x') < -q_k^{-1}\}, \qquad 1 \le j \le m_k - |\alpha|, \quad |\alpha| < m_k.$$

Let, further,

$$\Omega_{k,j}^{(\alpha)} = P_{k,j}^{(\alpha)} P_{k,j+1}^{(\alpha)} \cdots P_{k,m_k-|\alpha|}^{(\alpha)}, \qquad 0 \le j \le m_k - |\alpha|, \quad |\alpha| \le m_k,$$

so that $\Omega_k^{(\alpha)} = \Omega_{k,0}^{(\alpha)}$.

If a function $\psi_{k,j}^{(\alpha)}(x')$ is given on the set $\Gamma_{k,j}^{(\alpha)}$, and if for a function $u \in V$, after a possible alteration of the function $\Omega_{k,j}^{(\alpha)} u$ on a set of measure zero in Ω, the equality

$$\lim_{x_n \to 0} \Omega_{k,j}^{(\alpha)} u(x', x_n) = \psi_{k,j}^{(\alpha)}(x')$$

holds for almost all $x' \in \Gamma^{(\alpha)}_{k,j}$, then we shall write

$$\Omega^{(\alpha)}_{k,j} u\Big|_{\Gamma^{(\alpha)}_{k,j}} = \psi^{(\alpha)}_{k,j}(x'). \tag{2}$$

Definition 2. By the space $\dot{V}$ we shall mean the totality of all such functions $u \in V$ for which

$$\Omega^{(\alpha)}_{k,j} u\Big|_{\Gamma^{(\alpha)}_{k,j}} = 0, \qquad 1 \le j \le m_k - |\alpha|, \quad |\alpha| < m_k, \quad 1 \le k \le R, \tag{3}$$

and, in addition, $\partial^\nu u/\partial N^\nu|_{\Gamma_1} = 0$, $\nu = 0, \dots, m-1$, where N is the normal to the manifold Γ_1.

Remark. It is clear that if $\operatorname{mes} \Gamma^{(\alpha)}_{k,j} = 0$ for all $j = 1, \dots, m_k - |\alpha|$, $|\alpha| < m_k$, $k = 1, \dots, R$, then

$$\dot{V} = \{u \in V : \partial^\nu u/\partial N^\nu|_{\Gamma_1} = 0,\ \nu = 0, \dots, m-1\}.$$

In what follows we shall regard the space $\dot{V}$ as a normed space with norm $p(u)$.

3°. Properties of the space $\dot{V}$.

Theorem 1. Let $1 \le j \le m_k - |\alpha|$, $|\alpha| < m_k$, $1 \le k \le R$. Then for all $u \in \dot{V}$ the inequality

$$\left\| x_n^{\mu_{k,j}{}^{(\alpha)}(x')} \ln^{-1}(1 + x_n^{-1}) \Omega^{(\alpha)}_{k,j} u \right\|_{L_{q_k}(\Omega)} \le C \sum_{|\alpha| \le m_k} \left\| \Omega^{(\alpha)}_k u \right\|_{L_{q_k}(\Omega)} \tag{4}$$

holds, where C is a constant independent of u. If, moreover, $\varphi(x_n)$ is a bounded measurable function such that $\lim_{x_n \to 0} \varphi(x_n) = 0$, then the operator

$$A = \varphi(x_n) x_n^{\mu_{k,j}{}^{(\alpha)}(x')} \ln^{-1}(1 + x_n^{-1}) \Omega^{(\alpha)}_{k,j}$$

acts from the space $\dot{V}$ into the space $L_{q_k}(\Omega)$ and is completely continuous.

Remark. Theorem 1 can be strengthened; however, in that case its formulation becomes more cumbersome.

Theorem 2. The space $\dot{V}$ is a reflexive separable Banach space, and the set $C_0^{(\infty)}(\Omega)$ is dense in the space $\dot{V}$.

The proof of inequality (4) is carried out by successive application of the generalized Hardy inequality (see, for example, ([5])); here condition 1 is used. The complete continuity of the operator A is proved as follows. For any $\delta > 0$ the operator $A_\delta = \chi_\delta(x) A$, where $\chi_\delta(x) = 1$ if $x_n > \delta$, and $\chi_\delta(x) = 0$ if $x_n \le \delta$, is completely continuous by the theorem of V. I. Kondrashov ([6]). Further, from inequality (4) and the condition $\lim_{x_n \to 0} \varphi(x_n) = 0$ it follows that $\|A - A_\delta\| \to 0$ as $\delta \to 0$, and, consequently, the operator A is completely continuous.

The separability of the space $\dot{V}$ is obvious. Reflexivity follows from completeness, since the unit sphere in $\dot{V}$ is uniformly convex. To prove the completeness of $\dot{V}$, consider an arbitrary fundamental sequence $\{u_k\}$, $k = 1, 2, \ldots$. By a standard method one proves the existence of a function $u \in V$ for which $\partial^\nu u/\partial N^\nu|_{\Gamma_1} = 0$, $\nu = 0, \ldots, m-1$, and, moreover, $\lim_{k\to\infty} p(u - u_k) = 0$. Further, that

the fact that, for u, relations (3) are satisfied is established with the help of inequality (4).

The density of the set $C_0^{(\infty)}(\Omega)$ in $\dot{V}$ is proved on the basis of inequality (4) by the method of cut-off functions due to S. L. Sobolev.

4°. **The first boundary-value problem for degenerate elliptic equations.** In this section we shall confine ourselves to considering the first boundary-value problem for linear elliptic equations. However, using the Vishik–Browder theory, one could obtain, on the basis of Theorems 1 and 2, results on the solvability and unique solvability of the first boundary-value problem for quasilinear equations of elliptic type.

We shall assume that $q_1 = q_2 = \cdots = q_R = 2$, i.e., that $\dot{V}$ is a Hilbert space, and introduce the following notation. Let $0 \le j \le m_k - |\alpha|$, $|\alpha| \le m_k$, $1 \le k \le R$. Put

$$\mathcal{L}_{k,j}^{(\alpha)*} = P_{k,m_k-|\alpha|}^{(\alpha)*} P_{k,m_k-|\alpha|-1}^{(\alpha)*} \cdots P_{k,j+1}^{(\alpha)*} P_{k,j}^{(\alpha)*},$$

where

$$P_{k,j}^{(\alpha)*} u = -\frac{\partial}{\partial x_n} x_n^{\gamma_{k,j}{}^{(\alpha)}(x')} u \quad \text{for } |\alpha| < m_k,\ 0 \le j < m_k - |\alpha|,$$

and

$$P_{k,m_k-|\alpha|}^{(\alpha)*} u = (-1)^{|\alpha|} D_{x'}^{\alpha} x_n^{\gamma_{k,m_k-|\alpha|}{}^{(\alpha)}(x')} u.$$

For example,

$$\mathcal{L}_k^{(\alpha)*} = \mathcal{L}_{k,0}^{(\alpha)*} = P_{k,m_k-|\alpha|}^{(\alpha)*} \cdots P_{k,0}^{(\alpha)*}.$$

For convenience of notation we renumber the entire collection of operators

$$\{\mathcal{L}_{k,j}^{(\alpha)}\}, \qquad 0 \le j \le m_k - |\alpha|,\ |\alpha| \le m_k,\ 1 \le k \le R,$$

by means of a single index: $\mathcal{L}_1, \ldots, \mathcal{L}_{T_0}, \ldots, \mathcal{L}_T$, in such a way that the operator $\mathcal{L}_{k,j}^{(\alpha)}$ has number $\le T_0$ when $j = 0$ and $> T_0$ when $j > 0$. Next, to each operator

$\mathcal{L}_i$, $1 \le i \le T$, we assign a function $b_i(x)$ according to the following rule: if $\mathcal{L}_i = \mathcal{L}_{k,j}^{(\alpha)}$, then

$$b_i(x) \equiv 1, \text{ if } j = 0, \qquad b_i(x) = x_n^{\mu_{k,j}{}^{(\alpha)}(x')} \ln^{-1}(1 + x_n^{-1}), \quad \text{if } j > 0.$$

Introduce the following differential operators:

$$L_1 u = \sum_{i,s=1}^{T} \mathcal{L}_s^*(a_{is}^{(1)}(x)\mathcal{L}_i u), \qquad L_2 u = \sum_{i,s=1}^{T} \mathcal{L}_s^*(a_{is}^{(2)}(x)\mathcal{L}_i u),$$

$$Lu = L_1 u + L_2 u, \qquad L^* u = \sum_{i,s=1}^{T} \mathcal{L}_i^*((a_{is}^{(1)}(x) + a_{is}^{(2)}(x))\mathcal{L}_{su}).$$

Suppose the following conditions are satisfied:

$$1) \ \ |a_{is}^{(1)}(x)| + |a_{is}^{(2)}(x)| \le C b_i(x) b_s(x);$$

$$2) \ \ a_{is}^{(2)}(x) \equiv 0 \quad \text{when } \max(i,s) \le T_0;$$

$$3) \ \ \lim_{\delta \to 0} \sup_{x_n < \delta} (|a_{is}^{(2)}(x)|\, b_i^{-1}(x) b_s^{-1}(x)) = 0;$$

$$4) \quad \sum_{i,s=1}^{T} a_{is}^{(1)}(x) t_{it} s \ge \varepsilon_0 \sum_{i=1}^{T_0} t_i^2, \qquad \varepsilon_0 > 0.$$

Put

$$\widetilde{V} = \left\{ u \in V : \ \sum_{i,s=1}^{T} \left|(a_{is}^{(1)}(x) + a_{is}^{(2)}(x)) b_s^{-1}(x) \mathcal{L}_i u\right| \in L_2(\Omega) \right\},$$

$$\mathfrak{M}(u,v) = \sum_{i,s=1}^{T} \int_{\Omega} (a_{is}^{(1)}(x) + a_{is}^{(2)}(x)) \mathcal{L}_i u \, \mathcal{L}_s v \, dx, \qquad u \in \widetilde{V}, \ v \in \dot{V}.$$

Consider the equation

$$Lu = f, \qquad f \in \dot{V}^*, \tag{5}$$

and the first boundary-value problem for it

$$\mathcal{L}^{(\alpha)}_{k,j} u\big|_{\Gamma^{(\alpha)}_{k,j}} = \psi^{(\alpha)}_{k,j}(x'), \qquad 1 \le j \le m_k - |\alpha|,\ |\alpha| < m_k,\ 1 \le k \le R,$$

$$\partial^\nu u/\partial N^\nu\big|_{\Gamma_1} = \psi_\nu(\omega), \qquad \omega \in \Gamma_1, \quad \nu = 0, \dots, m-1. \tag{6}$$

Denote by $\mathfrak{A}$ the set of boundary conditions (6). The set of functions $u \in \dot{V}$ satisfying conditions (6) will be denoted by $\dot{V}_{\mathfrak{A}}$.

Definition 3. A function u is called a generalized solution (g.s.) of equation (5) if $u \in \dot{V}$ and $\mathfrak{M}(u, v) = \langle f, v\rangle$, $\forall v \in C^{(\infty)}_0(\Omega)$. A g.s. of the equations $L^* u = 0$ and $L_2 u = 0$ is defined analogously. We shall call a function u a g.s. of the first boundary-value problem (6) for equation (5) if u is a g.s. of equation (5) and $u \in \dot{V}_{\mathfrak{A}}$.

Denote by S, S^*, and S_2 the sets of g.s. of the equations $Lu = 0$, $L^* u = 0$, and $L_2 u = 0$, respectively, belonging to the space $\dot{V}$.

Definition 4. Let $\dot{V}_{\mathfrak{A}} \neq \emptyset$. If $\mathfrak{M}(u, v) = \langle f, v\rangle$, $\forall v \in S^*$, and $\forall u \in \dot{V}_{\mathfrak{A}}$, then we shall write $\{\mathfrak{A}, f\} \perp S^*$.

The following theorems are consequences of Theorems 1, 2 and of the theory of linear operators in Hilbert space.

Theorem 3. $\dim S = \dim S^* < \infty$.

Theorem 4. Let $\dot{V}_{\mathfrak{A}} \neq \emptyset$. Then a g.s. of the first boundary-value problem (6) for equation (5) in the space $\dot{V}$ exists if and only if $\{\mathfrak{A}, f\} \perp S^*$.

Theorem 5. Let $a^{(2)}_{is}(x) \equiv 0$, $\forall i, s = 1, \dots, T$. Let $\dot{V}_{\mathfrak{A}} \neq \emptyset$. Then in the space $\dot{V}$ there exists a unique g.s. of the first boundary-value problem (6) for equation (5) for any right-hand side $f \in \dot{V}^*$.

Theorem 6. If $a^{(1)}_{is}(x) = a^{(1)}_{si}(x)$, $a^{(2)}_{is}(x) = a^{(2)}_{si}(x)$, $\forall i, s = 1, \dots, T$, and $S_2 \neq \dot{V}$, then the eigenvalue problem $L_1 u - \lambda L_2 u = 0$, $u \in \dot{V}$, leads to a real, discrete, and finite-multiplicity spectrum with the only possible point of accumulation at infinity; moreover, the corresponding system of eigenfunctions forms an orthogonal basis in the space $\dot{V} \ominus S_2$.

Remark. Many concrete equations generally admit several representations in the form (5), and this means that for them there exist several spaces of type V and boundary-value problems of the form (6) corresponding to these spaces. We give a simple example ($R = 1,\ m = 2$).

$$Lu = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} x_n^{\gamma_{ij}(x')} \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{7}$$

where the function $\gamma_{ij}(x')$ is measurable on Γ_0' and $\sup_{x'\in\Gamma_0'}|\gamma_{ij}(x')| < \infty$. We restrict ourselves here to indicating 4 forms in which the operator L can be written,

$$Lu = x_n^{\omega}\frac{\partial}{\partial x_n}x_n^{\sigma}\frac{\partial}{\partial x_n}x_n^{\nu}\frac{\partial}{\partial x_n}x_n^{\sigma}\frac{\partial}{\partial x_n}x_n^{\omega}u + \sum_{i+j<2n}\frac{\partial^2}{\partial x_i\partial x_j}\,x_n^{\gamma_{ij}(x')}\frac{\partial^2 u}{\partial x_i \partial x_j};$$

1) $\omega = \sigma = 0, \quad \nu = \gamma_{nn}(x');$ 2) $\omega = 0, \quad \sigma = \gamma_{nn}(x') - 1, \quad \nu = 2 - \gamma_{nn}(x');$
3) $\omega = \gamma_{nn}(x') - 3, \quad \sigma = 3 - \gamma_{nn}(x'), \quad \nu = \gamma_{nn}(x');$ 4) $\omega = \gamma_{nn}(x') - 2, \quad \sigma = 0, \quad \nu = 4 - \gamma_{nn}(x');$

$$p(u) = \left(\left\|x_n^{\nu/2}\frac{\partial}{\partial x_n}x_n^{\sigma}\frac{\partial}{\partial x_n}x_n^{\omega}u\right\|^2_{L_2(\Omega)} + \sum_{i+j<2}\left\|x_n^{\gamma_{ij}(x')/2}\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|^2_{L_2(\Omega)}\right)^{1/2}. \quad (8)$$

Condition 1 is satisfied; therefore in all 4 cases boundary-value problems of the form (6) for equation (7) are posed as indicated above and are solved in spaces of type V, determined by the seminorms (8).

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Received
27 III 1970

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1. A. V. Bitsadze, *Equations of Mixed Type*, Publishing House of the Academy of Sciences of the USSR, 1959.
2. S. A. Tersenov, DAN, 115, No. 4, 670 (1957).
3. A. A. Vasharin, P. I. Lizorkin, DAN, 137, No. 5, 1015 (1961).
4. G. N. Yakovlev, *Differential Equations*, 4, No. 1, 147 (1968).
5. V. R. Portnov, DAN, 155, No. 4, 761 (1964).
6. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950.