ASYMPTOTIC DISTRIBUTION OF EIGENVALUES FOR DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

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Abstract

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ASYMPTOTIC DISTRIBUTION OF EIGENVALUES FOR DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

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Let G be a bounded domain in R^n. Let $P(D)$ be a symmetric differential operator with constant coefficients; here $D_j = -i\partial/\partial x_j$.

Suppose that the operator $P(D)$ satisfies the following conditions:

$$\lim_{|\xi|\to\infty} P(\xi) = +\infty, \tag{1}$$

$$\lim_{t\to\infty} \frac{S(P(\xi)=t)}{\mu(P(\xi)<t)} = 0, \tag{2}$$

where $S(P(\xi) = t)$ is the area of the surface $P(\xi) = t$; $\mu(P(\xi) < t)$ is the measure of the set of those $\xi \in R^n$ for which $P(\xi) < t$.

Denote by P_0 the operator $P(D)$ on functions from $C_0^\infty(G)$. Then P_0 is symmetric and, by virtue of condition (1), bounded from below. Let $\widetilde{P}$ be its self-adjoint Friedrichs extension. In this paper the asymptotic behavior of the eigenvalues of the operator $\widetilde{P}$ is studied. Previously this result was known only for elliptic operators. For the case of the Laplace operator and some other second-order operators it was obtained by Courant by the variational method. In the present paper, instead of Courant' s variational principle, the following theorem is used (^{1}):

Theorem 1. Let A be a self-adjoint operator bounded from below. Then the number of eigenvalues of the operator A that are less than t is equal to the maximal dimension of linear manifolds D such that

$$D \subseteq D(A), \qquad (Au, u) < t(u, u), \qquad u \in D.$$

Since $\widetilde{P}$ is the Friedrichs extension, in the present case the problem reduces to estimating the maximal dimension of linear manifolds D such that

$$D \subseteq C_0^\infty(G), \qquad (P(D)u, u) < t(u, u), \qquad u \in D.$$

1. Estimate of the dimension of manifolds D in the case of a cube

Let

$$Q_n = (x \in R^n;\ 0 \leq x_i \leq 2\pi;\ i = 1, 2, \dots, n).$$

Lemma 1. Let $D \subseteq C_0^\infty(Q_n)$ be a manifold of finite functions such that

$$(P(D)u, u) < t(u, u), \qquad u \in D.$$

Then

$$\dim D \leq \mu(P(\xi) < t). \tag{3}$$

Proof. Denote by Z^n the set of tuples $k = (k_1, \dots, k_n)$ of integers, and let $\beta = (\beta_1, \dots, \beta_n) \in I^n$, where I^n is the unit cube in R^n. The set of functions $e^{i\langle k+\beta, x\rangle}$, $k \in Z^n$, for any fixed β forms an orthonormal basis in $L_2(Q_n)$. Construct a self-adjoint extension of the operator P_0 in such a way that these functions are its

eigenfunctions with eigenvalues $P(k+\beta)$. Since $C_0^\infty(Q_n)$ will belong to the domain of definition of this self-adjoint extension, by Theorem 1,

$$\dim D \leq N(k+\beta;\ P(k+\beta) < t),$$

where $N(k+\beta;\ P(k+\beta) < t)$ is the number of points $k \in Z^n$ such that $P(k+\beta) < t$. Integrating this inequality over $\beta \in I^n$, we obtain

$$\dim D \leq \mu(P(\xi) < t).$$

Lemma 2. *The following relation holds*

$$N(k+\beta;\ P(k+\beta) < t) = \mu(P(\xi) < t) + O(S(P(\xi) = t)). \tag{4}$$

Proof. Denote the function $N(k+\beta;\ P(k+\beta) < t)$ by $N(\beta, t)$; the function $N(\beta, t)$ satisfies the relation

$$\int_{I^n} N(\beta, t)\, d\beta = \mu(P(\xi) < t). \tag{5}$$

Let $\beta_1, \beta_2 \in I^n$. Compare $N(\beta_1, t)$, $N(\beta_2, t)$. Consider a partition of the space R^n into cubes with vertices at the points $k + \beta_1$. Let $R(\beta_1, t)$ be the number of cubes lying entirely in the set $(\xi;\ P(\xi) < t)$, and let $T(\beta_1, t)$ be the number of cubes intersecting the surface $P(\xi) = t$.

In each cube lying entirely in the set $(\xi;\ P(\xi) < t)$ there will be a point of the form $k + \beta_2$. Therefore $N(\beta_2, t) \geq R(\beta_1, t)$, while

$$|N(\beta_1, t) - R(\beta_1, t)| \leq CT(\beta_1, t),$$

whence we obtain

$$N(\beta_2, t) \geq N(\beta_1, t) - CT(\beta_1, t).$$

For large t the surface $P(\xi) = t$ is a connected surface without singular points, since the set $(\xi; \operatorname{grad} P(\xi) = 0)$ consists of a finite number of connected pieces (2) and, by condition (1), they are all bounded in R^n. Then the number of cubes intersecting the surface $P(\xi) = t$ is estimated in terms of the area of this surface:

$$T(\beta_1, t) \leq C_1 S(P(\xi) = t).$$

Hence we obtain

$$N(\beta_2, t) \geq N(\beta_1, t) - C_2 S(P(\xi) = t).$$

Interchanging β_1 and β_2, we obtain the estimate

$$|N(\beta_2, t) - N(\beta_1, t)| \leq C_3 S(P(\xi) = t).$$

From this estimate and from (5) we obtain

$$N(\beta, t) = \mu(P(\xi) < t) + O(S(P(\xi) = t)).$$

Lemma 3. *The maximal dimension of subspaces* $D \subseteq C_0^\infty(Q_n)$ *such that*

$$(P(D)u, u) < t(u, u), \qquad u \in D,$$

is

$$\dim D = \mu(P(\xi) < t) + O(S(P(\xi) = t)). \tag{6}$$

Proof. The estimate of the dimension of the subspace D from above is given in Lemma 1. To prove the lemma it remains to estimate the dimension of D from below.

Consider the points with integer coordinates lying in the set $(\xi;\, P(\xi) < t)$. Take all exponentials of the form

$$e^{i\langle k,x\rangle}, \quad P(k) < t$$

and span the linear space $L(t)$ over them. Then

$$\dim L(t) = \mu(P(\xi) < t) + O(S(P(\xi) = t)),$$

$$(P(D)u, u) < t(u, u), \qquad u \in L(t). \tag{7}$$

Now in $L(t)$ we construct a subspace $L_0(t)$ consisting of functions having a zero of order m on the boundary Q_n, where m is the degree of the operator $P(D)$. Such functions belong to the domain of definition of the operator $\widehat{P}$. To construct the space $L_0(t)$, take an arbitrary point k_0 such that $P(k_0) < t$. Through the point k_0 draw a straight line parallel to the x_1-axis and take all integer points k on this line satisfying the condition $P(k) < t$. For each such point k consider the exponential $e^{i(k,x)}$ and span the linear space $L_1(k_0)$ over them. In the space $L_1(k_0)$ choose a subspace $L_1^0(k_0)$ such that

$$D_x^\alpha u\big|_{x_1=0,\, x_1=2\pi} = 0, \qquad |\alpha| \le m, \qquad u \in L_1^0(k_0).$$

The dimension of this space will be

$$\dim L_1^0(k_0) \ge \dim L_1(k_0) - 2(m+1).$$

Now take all such lines and, for each of them, construct an analogous space. The number of such lines is $O(S(P(\xi) = t))$. Taking the sum of all these spaces, we obtain a space $L_1(t)$ such that

$$D_x^\alpha u\big|_{x_1=0,\, x_1=2\pi} = 0, \qquad |\alpha| \le m, \qquad u \in L_1(t),$$

$$\dim L_1(t) = \mu(P(\xi) < t) + O(S(P(\xi) = t)).$$

Construct the spaces $L_2(t), \dots, L_n(t)$ corresponding to the variables $x_2, \dots, x_n$. Put

$$L_0(t) = L_1(t) \cap L_2(t) \cap \dots \cap L_n(t).$$

Then

$$D_x^\alpha u\big|_{\partial Q_n} = 0, \qquad |\alpha| \le m, \qquad u \in L_0(t),$$

$$\dim L_0(t) = \mu(P(\xi) < t) + O(S(P(\xi) = t)).$$

This proves the lemma. In the case of a cube Q with side of arbitrary length, by a change of variables we obtain the formula

$$\dim \mathcal{D} = (2\pi)^{-n}\mu(Q)\mu(P(\xi) < t) + O(S(\partial Q) \cdot S(P(\xi) = t)).$$

2. Asymptotic distribution in the case of an arbitrary domain G. Let G be a bounded domain in R^n. Denote by $N(t)$ the number of eigenvalues of the operator $\hat{P}$ less than t. We estimate $N(t)$ from below. Construct pairwise nonintersecting cubes Q^j contained in G, and for each cube construct a manifold $L_0^j(t)$ such that

$$L_0^j(t) \subseteq C_0^\infty(Q^j); \qquad (P(D)u, u) < t(u, u), \qquad u \in L_0^j(t),$$

$$\dim L_0^j(t) = (2\pi)^{-n}\mu(Q^j)\mu(P(\xi) < t) + O(S(\partial Q^j)S(P(\xi) = t)).$$

The sum of all these manifolds gives a manifold $L_0(t) \subseteq C_0^\infty(G)$, on which

$$(P(D)u, u) < t(u, u), \qquad u \in L_0(t). \tag{8}$$

From Theorem 1 it follows that

$$N(t) \ge \dim L_0(t) = \sum_j \dim L_0^j(t) \ge$$

$$\ge (2\pi)^{-n}\mu(P(\xi) < t)\sum_j \mu(Q^j) - CS(P(\xi) = t)\sum_j S(\partial Q^j). \tag{9}$$

From (9), taking (2) into account, we obtain that

$$\liminf_{t\to\infty} \frac{N(t)}{(2\pi)^{-n}\mu(G)\mu(P(\xi) < t)} \ge \frac{\sum_j \mu(Q^j)}{\mu(G)}$$

and hence

$$\liminf_{t\to\infty} \frac{N(t)}{(2\pi)^{-n}\mu(G)\mu(P(\xi)<t)} \geq 1. \tag{10}$$

Now let us estimate $N(t)$ from above. Take a cube $Q \supseteq G$ and construct cubes Q^j so that

$$Q^j \subseteq Q \setminus G, \qquad Q^j \cap Q^i = \emptyset, \quad i \neq j.$$

In each Q^j construct a manifold $L_0^j(t)$ such that

$$L_0^j(t) \subseteq C_0^\infty(Q^j), \qquad (P(D)u,u) < t(u,u), \qquad u \in L_0^j(t),$$

$$\dim L_0^j(t) = (2\pi)^{-n}\mu(Q^j)\mu(P(\xi)<t) + O\big(S(\partial Q^j)S(P(\xi)=t)\big).$$

Let now $D \subseteq C_0^\infty(G)$

$$(P(D)u,u) < t(u,u), \qquad u \in D; \qquad \dim D = N(t).$$

Take the sum of the manifolds D and $L_0^j(t)$. On this sum the estimate

$$(P(D)u,u) < t(u,u)$$

holds. From Lemma 1 we have

$$N(t) + \sum_j \dim L_0^j(t) \leq (2\pi)^{-n}\mu(Q)\mu(P(\xi)<t)$$

or

$$N(t) \leq (2\pi)^{-n}\mu(P(\xi)<t)\left(\mu(Q) - \sum_j \mu(Q^j)\right) + CS(P(\xi)=t)\sum_j S(\partial Q^j). \tag{11}$$

From (11) we obtain

$$\limsup_{t\to\infty} \frac{N(t)}{(2\pi)^{-n}\mu(P(\xi)<t)\mu(G)} \leq \frac{\mu(Q) - \sum_j \mu(Q^j)}{\mu(G)}$$

and hence, when $\mu(\partial G) = 0$,

$$\limsup_{t\to\infty} \frac{N(t)}{(2\pi)^{-n}\mu(P(\xi) < t)\mu(G)} \leq 1. \tag{12}$$

From (10), (12) it follows:

Theorem 2. If conditions (1), (2) are satisfied and $\mu(\partial G) = 0$, then the function $N(t)$ has the asymptotic

$$N(t) \sim (2\pi)^{-n}\mu(G)\mu(P(\xi) < t).$$

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Note: Figure translations are in progress. See original paper for figures.

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