ON THE NORMAL EULER NUMBERS OF THE PROJECTIVE PLANE AND THE KLEIN BOTTLE IN FOUR-DIMENSIONAL EUCLIDEAN SPACE

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 513.832/835

MATHEMATICS

V. A. ROKHLIN

ON THE NORMAL EULER NUMBERS OF THE PROJECTIVE PLANE AND THE KLEIN BOTTLE IN FOUR-DIMENSIONAL EUCLIDEAN SPACE

(Presented by Academician L. S. Pontryagin on 15 VII 1969)

1. Formulation of the results. In this note, by manifolds we mean smooth manifolds, and such terms as embedding, immersion, and isotopy are understood in the differential-topological sense. If f is an immersion of a closed n-dimensional manifold M in the oriented Euclidean space $\mathbf{R}^{2n}$, then the normal Euler number $x(f) \in \mathbf{Z}$ is defined. Let us recall that it changes sign when $\mathbf{R}^{2n}$ changes orientation, and is equal to zero for odd n, and also for even n if the manifold M is orientable and f is an embedding. By N_k we denote the nonorientable connected closed two-dimensional manifold with Euler characteristic $2-k$ (the sphere with k Möbius bands).

In 1941 Whitney (^{1}) published a theorem according to which $x(f) \equiv 2k \pmod 4$ for every embedding $f : N_k \to \mathbf{R}^4$. In the same paper he formulated the problem of a complete description of the values $x(f)$ corresponding to all possible embeddings $f : N_k \to \mathbf{R}^4$, and put forward the supposition that these are the values $-2k, -2k+4, \dots, 2k-4, 2k$. That these values are realized is not difficult to show: for the projective plane $\mathbf{RP}^2 = N_1$, an embedding in $\mathbf{R}^4$ with $x = \pm 2$ is well known (and is described, for example, in (^{2}), p. 295); for N_k with arbitrary k, realizing embeddings may be constructed by using the decomposition $N_k = \mathbf{RP}^2 \# \cdots \# \mathbf{RP}^2$ (k summands), embedding the summands in $\mathbf{R}^4$ with $x = \pm 2$, and joining them by tubes. For example, in the case of the Klein bottle $Kl = N_2 = \mathbf{RP}^2 \# \mathbf{RP}^2$, for the summands there are four combinations of the values $x = \pm 2$ ($-2,-2$; $-2,2$; $2,-2$; $2,2$), leading to embeddings $Kl \to \mathbf{R}^4$ with $x = -4, 0, 0, 4$.

In 1964 Mahowald (^{3}) published a multidimensional generalization of Whitney's theorem. For a connected closed manifold M of dimension n, define the number $\omega(M)$ by the formula

$$\omega(M) = 0, \text{ if } \overline{w}_1(M)\,\overline{w}_{n-1}(M) = 0, \quad \omega(M) = 1, \text{ if } \overline{w}_1(M)\,\overline{w}_{n-1}(M) \neq 0,$$

where $\overline{w}_i$ are the normal Stiefel–Whitney classes. Obviously, $\omega(N_k) \equiv k \pmod 2$, and, according to a theorem of Massey (^{4}), $\omega(M) = 0$ if n is not a power of two. Mahowald proved that if n is even and $f : M \to \mathbf{R}^{2n}$ is an embedding, then $x(f) \equiv 2\omega(M) \pmod 4$. As Massey observed (^{5}), for $n > 2$ this theorem is easily inverted: for every nonorientable connected closed manifold M of even dimension $n > 2$ and every integer x satisfying the congruence $x \equiv 2\omega(M) \pmod 4$, there exists an embedding $f : M \to \mathbf{R}^{2n}$ with $x(f) = x$.

The present note is devoted to Whitney' s original problem ($n = 2$). It is proved here that for every embedding $\mathbf{RP}^2 \to \mathbf{R}^4$ the normal Euler number is congruent modulo 16 to one of the numbers $-2, 2$, and for every embedding $Kl \to \mathbf{R}^4$ the normal Euler number is congruent modulo 16 to one of the numbers $-4, 0, 4$.

The proof is based on a geometric construction connecting Whitney' s problem with the well-known problem of realizing two-dimensional homology classes in a four-dimensional manifold by spheres.

The same method can also be applied to the surfaces N_k with $k > 2$, but for them it does not lead to new results.

2. Proof of the relation $x(f) \equiv \pm 2 \pmod{16}$ **for an embedding** $f : \mathbb{R}P^2 \to \mathbb{R}^4$**.** Cut the projective plane $\mathbb{R}P^2$ in the usual way into a disk K and a Möbius band L, and embed L in the standard way in $\mathbb{R}^3$. Composing this embedding with the inclusion $\mathbb{R}^3 \subset \mathbb{R}^4$, we obtain an embedding $\varphi : L \to \mathbb{R}^4$, under which the band $\varphi(L)$ meets, by its boundary C, the boundary $\partial T = S^1 \times S^2$ of the tubular neighborhood $T = S^1 \times D^3$ of the circle S^1 in $\mathbb{R}^4$ orthogonally.

An embedding $f : \mathbb{R}P^2 \to \mathbb{R}^4$ is called **special** if $f|_L = \varphi$ and $f(K) \subset \mathbb{R}^4 \setminus \operatorname{Int} T$. It is not difficult to show that every embedding $\mathbb{R}P^2 \to \mathbb{R}^4$ is isotopic to a special one. Therefore, in Whitney' s problem for the projective plane one may restrict oneself to special embeddings.

Construct in $\mathbb{R}^4$, on the band $\varphi(L)$, some nonsingular normal vector field u, and denote by v its restriction to C. If $f : \mathbb{R}P^2 \to \mathbb{R}^4$ is a special embedding, then the field v determines at the center of the disk $f(K)$, for which the circle C serves as boundary, a singularity whose index is $x(f)$. In particular, the field v, considered as a normal vector field on C in ∂T, is independent, up to homotopy, of the choice of the field u. Adding a point to $\mathbb{R}^4$ to obtain S^4, and thereby turning $\mathbb{R}^4 \setminus \operatorname{Int} T$ into $S^4 \setminus \operatorname{Int} T = D^2 \times S^2$, we arrive at the following reformulation of Whitney' s problem for the projective plane: on the boundary $S^1 \times S^2$ of the manifold $D^2 \times S^2$ lies the (described above) circle C, equipped with the (described above) normal vector field v; what indices does this field determine on all possible disks spanning (smoothly and without self-intersections) C in $D^2 \times S^2$ orthogonally to the boundary? We shall call such disks **normal**, and the index determined by the field v on a normal disk X will be denoted by $i(X)$.

The orientation of the space $\mathbb{R}^4$ determines an orientation of the manifold $D^2 \times S^2$. We orient S^2, and thereby all normal disks. If X, Y are normal disks, then the difference $\delta(X, Y) \in H_2(D^2 \times S^2) = \mathbb{Z}$ is defined with the usual properties;

in particular, $\delta(X,Y) + \delta(Y,Z) = \delta(X,Z)$. If one brings into consideration the double $S^2 \times S^2$ of the manifold $D^2 \times S^2$ and realizes X in one half and Y in the other, then in $S^2 \times S^2$ an embedded sphere is formed from X and Y, representing the homology class $2\xi + \delta(X,Y)\eta$, where ξ, η are the natural generators of the group $H_2(S^2 \times S^2)$. The self-intersection index of this class is equal to $4\delta(X,Y)$, and it is also equal to $i(Y) - i(X)$.

Since, according to § 1, there exist embeddings $\mathbb{R}P^2 \to \mathbb{R}^4$ with normal Euler numbers -2 and 2, there exist normal disks X_0, X_1 with $i(X_0) = -2$ and $i(X_1) = 2$. Since $i(X_1) - i(X_0) = 4\delta(X_0, X_1)$, we have $\delta(X_0, X_1) = 1$, and therefore $\delta(X_0, X) = \delta(X_1, X) + 1$ for any normal disk X. The last formula shows that one of the numbers $\delta(X_0, X)$, $\delta(X_1, X)$ is even, and hence one of the two classes $2\xi + \delta(X_j, X)\eta \in H_2(S^2 \times S^2)$ is divisible by 2. Since these classes are realized by embedded spheres, by the theorem of Kervaire-Milnor (^{6}), one of the numbers $\delta(X_0, X)$, $\delta(X_1, X)$ is divisible by 4. Thus, $i(X) \equiv \pm 2 \pmod{16}$.

3. Proof of the relation $x(f) \equiv 0, \pm 4 \pmod{16}$ for an embedding $f : Kl \to \mathbb{R}^4$. This proof is analogous to the preceding one, but somewhat more complicated. The Möbius band, i.e. the punctured projective plane, is replaced by the punctured Klein bottle. The middle line of the new band is a bouquet of two circles. There is a standard embedding of this band in S^4, under which the band is placed in the standard neighborhood T of the standard bouquet of two circles and meets ∂T orthogonally by its boundary C. Every embedding $Kl \to S^4$ is isotopic to a special embedding, agreeing on the band with the standard one and carrying the disk that completes the band to Kl into $V = S^4 \setminus \operatorname{Int} T$. As in § 1, the band determines on C in ∂T, uniquely up to homotopy, a normal vector ···

field, and the Whitney problem for the Klein bottle turns out to be equivalent to the following problem: on the boundary ∂V of the manifold V there lies the (specified) circle C, endowed with the specified vector field; what $i(X)$ does this field determine on all possible normal disks X, i.e., disks stretched on C in V smoothly, without self-intersections, and orthogonally to ∂V?

The manifold V is the boundary connected sum of two copies of the manifold $D^2 \times S^2$, i.e., it is obtained from them by identifying a three-dimensional ball taken on the boundary of the first copy with its double on the boundary of the second copy. In particular,

$$\partial V = S^1 \times S^2 \# S^1 \times S^2.$$

We adopt the same conventions on orientations as in § 2. As there, for normal disks X, Y a difference $\delta(X,Y)$ with the usual properties is defined; however now $\delta(X,Y) \in H_2(V) = \mathbf{Z} + \mathbf{Z}$, so that $\delta(X,Y)$ is a pair of integers

$$\delta_1 = \delta_1(X,Y), \qquad \delta_2 = \delta_2(X,Y).$$

In the doubling $S^2 \times S^2 \# S^2 \times S^2$ of the manifold V, the disks X, Y determine an embedded sphere realizing the homology class

$$2(\xi_1+\xi_2)+\delta_1\eta_1+\delta_2\eta_2,$$

where $\xi_1, \eta_1; \xi_2, \eta_2$ are the natural generators of the group

$$H_2(S^2\times S^2\# S^2\times S^2).$$

Computing, as in § 2, the self-intersection index of this class in two ways, we see that

$$i(Y)-i(X)=4(\delta_1+\delta_2).$$

To the four embeddings $Kl\to\mathbf{R}^4$ described in item 1 there correspond normal disks

$$X_{00},\ X_{01},\ X_{10},\ X_{11}$$

with

$$i(X_{00})=-4,\qquad i(X_{01})=i(X_{10})=0,\qquad i(X_{11})=4$$

and with

$$\delta_1(X_{pq},X_{rs})=r-p,\qquad \delta_2(X_{pq},X_{rs})=s-q$$
$$(p,q,r,s=0,1).$$

Hence, by virtue of

$$\delta_j(X_{pq},X)=\delta_j(X_{rs},X)+\delta_j(X_{pq},X_{rs})\qquad (j=1,2),$$

for every normal disk X there exists a disk X_{pq} with even

$$\delta_1(X_{pq},X),\quad \delta_2(X_{pq},X).$$

By virtue of the already cited Kervaire-Milnor theorem, the self-intersection index of the corresponding class

$$2(\xi_1+\xi_2)+\delta_1\eta_1+\delta_2\eta_2$$

is divisible by 16, so that

$$i(X)-i(X_{pq})\equiv 0\pmod{16}.$$

Thus

$$i(X)=0,\ \pm 4\pmod{16}\ *.$$

Leningrad State University
named after A. A. Zhdanov

Received
10 VII 1969

REFERENCES

1. H. Whitney, *Lectures in Topology*, 1941.
2. D. Hilbert, S. Cohn-Vossen, *Visual Geometry*, 1936.
3. M. Mahowald, Pacific. J. Math., **14**, No. 4, 1335 (1964).
4. W. S. Massey, Proc. Am. Math. Soc., **13**, No. 6, 938 (1962).
5. B. D. Malyi, Matem. zametki, **5**, No. 1 (1969).
6. M. A. Kervaire, J. W. Milnor, Proc. Nat. Acad. Sci. U.S.A., **47**, No. 10, 1651 (1961).
7. W. F. Massey, Pacific. J. Math., **31**, 143 (1969).

---

* **Note added in proof.** 1. Recently Massey (^{7}) proved the Whitney conjecture in full scope: if

$$f : N_k \to \mathbf{R}^4$$

is an embedding, then $\varkappa(f)$ coincides with one of the numbers

$$-2k,\ -2k+4,\ \dots,\ 2k.$$

2. At present the author has proved that if X is a smooth closed 4-dimensional manifold with $H_1(X) = 0$, second Betti number b, and signature σ, then the genus of a smooth surface realizing an element ξ of the group $H_2(X)$ divisible by $m \geq 2$ cannot be less than

$$|(m+1)\xi\xi/6m - \sigma/2| - b/2$$

($\xi\xi$ is the self-intersection index). In particular, in CP^2 smooth spheres realize only the generators and doubled generators of $H_2(P_2C)$, while in $S^2 \times S^2$ a class $p\xi + q\eta$ with $p \neq 0,\ q \neq 0$ is not realized by a smooth sphere unless p and q are mutually prime (ξ and η are the natural generators of $H_2(S^2 \times S^2)$). An analogous estimate exists for the genus of a nonorientable smooth surface in X, and the Whitney conjecture is nothing other than a special case of this general estimate ($X = S^4$). However, for RP^2 the validity of the Whitney conjecture follows in an obvious way from the theorem just formulated for $S^2 \times S^2$ and the reduction of item 2.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.