

# FINITE PLANE DEFORMATIONS OF AN ELASTIC ISOTROPIC BODY

THEORY OF ELASTICITY

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## Abstract

## Full Text

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*THEORY OF ELASTICITY*

**G. S. TARAS' EV**

# FINITE PLANE DEFORMATIONS OF AN ELASTIC ISOTROPIC BODY

*(Presented by Academician L. I. Sedov on 25 III 1970)*

For the plane deformed state of an ideally elastic isotropic body, resolving equations are given in the complex coordinates of the initial state. In contrast to the work <sup>(1)</sup>, no restrictions of a geometric or physical nature are imposed.

**1. Geometric relations.** Let  $W$  be the complex displacement of an individual particle of the body under plane deformation. Then the correspondence between the values  $dz = ds_0 e^{i\beta_0}$ ,  $d\eta = ds e^{i\beta}$  of an element of this particle in the initial and final states is determined by the relation

$$d\eta = dz + dW, \quad (1)$$

whence

$$\lambda e^{i\beta} = e^{i\beta_0}(1 + W_z) + W_{\bar{z}} e^{-i\beta_0}, \quad (2)$$

where  $\lambda = ds/ds_0$ , and the indices  $z, \bar{z}$  denote partial derivatives with respect to the Lagrangian complex coordinates.

If  $\beta_0 = \theta_0$ ,  $\beta_0 = \theta_0 + \pi/2$  are the angles formed by the first and third principal directions of deformation in the initial state with the axis  $x_1$ , and  $\beta = \theta_0 + \omega$ ,  $\beta = \theta_0 + \omega + \pi/2$  are the same in the deformed state, then from formulas (2) one may obtain

$$\lambda_1 e^{i\theta_0} e^{i\omega} = e^{i\theta_0}(1 + W_z) + W_{\bar{z}} e^{-i\theta_0}, \quad \lambda_3 e^{i\theta_0} e^{i\omega} = e^{i\theta_0}(1 + W_z) - W_{\bar{z}} e^{-i\theta_0},$$

where  $\lambda_1, \lambda_3$  correspond to the principal directions of deformation, and  $\omega$  represents, in the general case, the finite angle of rotation of the principal axes of deformation in the plane  $(x_1, x_3)$ .

Adding and subtracting the last formulas, we obtain

$$1 + W_z = \frac{1}{2}(\lambda_1 + \lambda_3)e^{i\omega} = e^{v/2} \operatorname{ch}(\varepsilon/2)e^{i\omega}, \quad v = \ln \lambda_1 + \ln \lambda_3,$$

$$W_{\bar{z}} = \frac{1}{2}(\lambda_1 - \lambda_3)e^{2i\theta_0}e^{i\omega} = e^{v/2} \operatorname{sh}(\varepsilon/2)e^{2i\theta_0}e^{i\omega}, \quad \varepsilon = \ln \lambda_1 - \ln \lambda_3, \quad (3)$$

where  $v, \varepsilon$  are the plane invariants of the Hencky tensor <sup>(2)</sup>.

From this, in particular, follows the equality

$$W_{\bar{z}} = \operatorname{th}(\varepsilon/2)e^{2i\theta_0}(1 + W_z), \quad (4)$$

by means of which the relation between the displacement function and the stress function is established.

Eliminating the derivatives of the displacement function from equations (3) and passing in the resulting relation to conjugate values, we obtain an algebraic system of two equations with respect to the quantities  $\omega_z, \omega_{\bar{z}}$  with a determinant different from zero. Solving this system, we find

$$2i\omega_z = e^{-v} [-(e^v \operatorname{sh} \varepsilon e^{-2i\theta_0})_{\bar{z}} + (e^v \operatorname{ch} \varepsilon)_z + 2ie^v(\operatorname{ch} \varepsilon - 1)\theta_z].$$

$$2i\omega_{\bar{z}} = e^{-v} [(e^v \operatorname{sh} \varepsilon e^{2i\theta_0})_z - (e^v \operatorname{sh} \varepsilon)_{\bar{z}} + 2ie^v(\operatorname{ch} \varepsilon - 1)\theta_{\bar{z}}]. \quad (5)$$

Eliminating from these equalities the derivatives of the angle of rotation, we obtain the compatibility condition for the deformation parameters

$$\operatorname{Re}\{[e^{-v}(e^v \operatorname{sh} \varepsilon e^{2i\theta_0})_z]_{\bar{z}} - [e^{-v}(e^v \operatorname{ch} \varepsilon)_{\bar{z}}]_z + 2i[(\operatorname{ch} \varepsilon - 1)\theta_{\bar{z}}]_z\} = 0. \quad (6)$$

In deriving the equilibrium equations, expressions will be needed for the derivatives of the displacement function with respect to the complex coordinates  $\eta = z + \bar{W}$ ,  $\bar{\eta} = \bar{z} + W$ , which characterize the position of particles in the deformed state.

For an arbitrary function  $\xi = \xi[z(\eta, \bar{\eta}), \bar{z}(\eta, \bar{\eta})]$  the formulas

$$\begin{aligned} \xi_\eta &= \xi_z z_\eta + \xi_{\bar{z}} \bar{z}_\eta = (1 - W_\eta)\xi_z - \bar{W}_\eta \xi_{\bar{z}}, \\ \xi_{\bar{\eta}} &= \xi_z z_{\bar{\eta}} + \xi_{\bar{z}} \bar{z}_{\bar{\eta}} = -W_{\bar{\eta}} \xi_z + (1 - \bar{W}_{\bar{\eta}})\xi_{\bar{z}}. \end{aligned}$$

hold. Setting in these formulas first  $\xi = W$ , and then  $\xi = \bar{W}$ , we obtain an algebraic system of equations for the derivatives of the displacement function  $W$  with respect to the variables  $\eta, \bar{\eta}$ . The solution of this system is

$$1 - W_\eta = e^{-\nu/2} \operatorname{ch}(\varepsilon/2)e^{-i\omega}, \quad \bar{W}_\eta = e^{-\nu/2} \operatorname{sh}(\varepsilon/2)e^{-2i\theta_0}e^{-i\omega}. \quad (7)$$

**2. Physical laws.** We shall assume, as in [3], that there exists an elastic stress potential, calculated per unit volume of the undeformed particle of the body. The increment of the potential is represented in terms of the true principal stresses and the logarithmic strains  $h_k = \ln \lambda_k$  [2] in the form [3, 4]

$$\delta A = (1 + \Delta)\sigma_k \delta h_k,$$

where  $\Delta$  is the relative change in volume of the particle.

For plane strain  $h_2 = 0$ , and the increment of the potential takes the form

$$\delta A = e^\nu \Sigma \delta \nu + e^\nu T \delta \varepsilon,$$

where  $\Sigma = \frac{1}{2}(\sigma_1 + \sigma_3)$ ,  $T = \frac{1}{2}(\sigma_1 - \sigma_3)$ .

Since  $\delta A$  is an exact differential, it follows that

$$e^\nu \Sigma = \partial A / \partial \nu, \quad e^\nu T = \partial A / \partial \varepsilon. \quad (8)$$

If, in particular, the stress potential is represented by analytic series in even powers of the characteristics  $h_0, h$  of change in volume and change in shape [3],

$$A = K \left( \frac{1}{2} h_0^2 + \frac{1}{4} k_3 h_0^4 + \dots \right) + 6G \left( \frac{1}{2} h^2 + \frac{1}{4} g_3 h^4 + \dots \right),$$

then

$$\begin{aligned} e^\nu \Sigma &= \frac{G}{1-2\nu} \left[ \nu + \frac{1-2\nu}{G} \left( K k_3 + \frac{G g_3}{54} \right) \nu^3 + (1-2\nu) \frac{g_3}{18} \nu \varepsilon^2 + \dots \right], \\ e^\nu T &= G \left[ \varepsilon + \frac{g_3}{6} \varepsilon^3 + \frac{g_3}{18} \varepsilon \nu^2 + \dots \right], \quad \nu = \frac{1}{2} \frac{3K-2G}{3K+G}. \end{aligned} \quad (9)$$

The inverse relations, determined according to the rules for inversion of analytic series, have the form

$$\begin{aligned} \nu &= (1-2\nu)t_0 - (1-2\nu)^3 \left[ \frac{K}{G} k_3 + \frac{1}{54} g_3 \right] t_0^3 - \frac{1}{18} (1-2\nu)^2 g_3 t_0 t^2 + \dots, \\ \varepsilon &= t - \frac{1}{18} (1-2\nu)^2 g_3 t_0^2 - \frac{1}{6} g_2 t^3 + \dots, \quad t_0 = e^\nu \Sigma / G, \quad t = e^\nu T / G. \end{aligned} \quad (10)$$

**3. Stress function.** The equilibrium equations in coordinates of the final state, in complex form, have the classical form

$$\Sigma_{\bar{\eta}} + (Te^{2i\theta})_{\eta} + \frac{1}{2}\rho\bar{F} = 0, \quad (11)$$

where  $\rho, F$  are the density and the complex body force.

Passing in equation (11) from differentiation with respect to the coordinates  $\eta, \bar{\eta}$  to differentiation with respect to the initial coordinates  $z, \bar{z}$  and using

in this case, from formulas (7), we find

$$\begin{aligned} & \text{ch}(\varepsilon/2)\Sigma_z - \text{sh}(\varepsilon/2)e^{2i\theta_0}\bar{\Sigma}_z + \text{ch}(\varepsilon/2)(Te^{2i\theta_0})_z + 2\text{ch}(\varepsilon/2)Te^{2i\theta_0}i\omega_z \\ & - \frac{1}{2}\text{sh}\varepsilon/2e^{-2i\theta_0}\overline{(Te^{2i\theta_0})_z} + 2\text{sh}(\varepsilon/2)Ti\omega_{\bar{z}} + \frac{1}{2}e^{\nu/2}\rho Fe^{-i\omega} = 0. \end{aligned}$$

First multiply this equation by  $\text{ch}(\varepsilon/2)$ , and then, after first passing to the conjugate expression of this equation, by  $\text{sh}(\varepsilon/2)e^{2i\theta_0}$ . Adding the equations obtained and using formulas (5), we find

$$\begin{aligned} & \text{ch}\varepsilon\{\Sigma_z + (Te^{2i\theta_0})_z + Te^{2i\theta_0}v_z - T\varepsilon_z\} - \text{sh}\varepsilon e^{2i\theta_0}\{\Sigma_z + (Te^{-2i\theta_0})_{\bar{z}} \\ & + Te^{-2i\theta_0}v_{\bar{z}} - T\varepsilon_{\bar{z}}\} + \frac{1}{2}\rho e^{\nu/2}[F\text{ch}(\varepsilon/2)e^{-i\omega} - \bar{F}\text{sh}(\varepsilon/2)e^{2i\theta_0}e^{i\omega}] = 0. \end{aligned}$$

If in this equation we pass to the conjugate expression, we obtain a system of two algebraic equations with respect to the conjugate differential operators. Solving this system and multiplying the result by  $e^{\nu}$ , we obtain

$$\begin{aligned} & (e^{\nu}\Sigma)_{\bar{z}} + (e^{\nu}Te^{2i\theta_0})_z - (e^{\nu}\Sigma v_{\bar{z}} + e^{\nu}T\varepsilon_z) \\ & = \rho_0\{\text{sh}(\varepsilon/2)e^{-2i\theta_0}(F\text{ch}(\varepsilon/2)e^{-i\omega} - \bar{F}\text{sh}(\varepsilon/2)e^{-2i\theta_0}e^{i\omega}) \\ & - \text{ch}\varepsilon(\bar{F}\text{ch}(\varepsilon/2)e^{i\omega} - F\text{sh}(\varepsilon/2)e^{2i\theta_0}e^{-i\omega})\}. \end{aligned} \quad (12)$$

Noting that, by virtue of relations (8), the expression

$$e^{\nu}\Sigma v_{\bar{z}} + e^{\nu}T\varepsilon_z = A_{\bar{z}}$$

is the partial derivative of the stress potential with respect to the complex coordinate, and omitting the body forces in (12), we arrive at the equilibrium equation in complex form, whose structure resembles the classical one,

$$(e^{\nu}\Sigma - A)_{\bar{z}} + (e^{\nu}Te^{2i\theta_0})_z = 0. \quad (13)$$

The latter equation can be satisfied by introducing a generalized stress function

$$e^\nu \Sigma = 2pU_{z\bar{z}} + A, \quad e^\nu T e^{2i\theta_0} = -2pU_{z\bar{z}}. \quad (14)$$

Here  $p$  is a characteristic constant stress.

Substituting the value of  $e^\nu e^{2i\theta_0}$  into the compatibility condition (6), we arrive at a nonlinear differential equation with respect to the generalized stress function,

$$\begin{aligned} \operatorname{Re} \left\{ \left[ e^{-\nu} \left( p \frac{\operatorname{sh} \varepsilon}{T} U_{z\bar{z}} \right) \right]_z + \frac{1}{2} [e^{-\nu} (e^\nu \operatorname{ch} \varepsilon)]_z \right. \\ \left. + \frac{p^2}{(e^\nu T)^2} (\operatorname{ch} \varepsilon)_z (U_{zz} U_{z\bar{z}} - U_{z\bar{z}} U_{zz\bar{z}}) \right\} = 0. \end{aligned} \quad (15)$$

This equation has been obtained without any restrictions of a geometrical or physical character, under the sole assumption of the existence of the potential of the internal forces. Using in this equation specific physical laws, for example laws (9) or others, one can obtain various versions of the governing equation of the theory of a plane deformed state.

Using the last relation from equations (14) in formula (4), we obtain a relation between the complex displacement function and the generalized stress function,

$$W_z = -2p \frac{\operatorname{th}(\varepsilon/2)}{e^\nu T} U_{z\bar{z}} (1 + W_z). \quad (16)$$

This relation is also valid for arbitrary elastic physical laws. From it one can obtain, by successive approximations, the geometrical boundary conditions.

**4. Static boundary conditions.** If  $p\sigma_n$ ,  $p\tau_n$  are the true normal and tangential stresses on the deformed contour with normal  $n(\cos \alpha, \sin \alpha)$ , then

$$\Sigma + T e^{2i(\theta-\alpha)} = p(\sigma_n + i\tau_n). \quad (17)$$

For a contour fiber ( $\lambda = \lambda_t$ ), from formula (2) we find

$$\lambda_t e^{i\alpha} = e^{i\alpha_0} (1 + W_z) - W_{\bar{z}} e^{-i\alpha_0}, \quad (18)$$

where  $\alpha_0$  is the angle between the normal to the initial contour and the  $x_1$ -axis.

Squaring (18) and substituting into (17), we obtain

$$\begin{aligned} p\sigma_n \lambda_t^2 &= \lambda_t^2 \Sigma + e^\nu T [-\operatorname{sh} \varepsilon + \operatorname{ch} \varepsilon \cos 2(\theta_0 - \alpha_0)], \\ p\tau_n \lambda_t^2 &= e^\nu T \sin 2(\theta_0 - \alpha_0). \end{aligned} \quad (19)$$

Taking into account that on the contour the geometric relation

$$ie^{-i\alpha_0}dU_z/ds_0 = U_{zz}e^{-2i\alpha_0} - U_{z\bar{z}}$$

holds, with the aid of formulas (14), (18), (19) we obtain the static boundary conditions for the stress function

$$dU_z/ds_0 = \frac{1}{2}e^\nu(\sigma_n + i\tau_n \operatorname{ch} \varepsilon)ie^{i\alpha_0} - p\tau_n \operatorname{sh} \varepsilon/TU_z e^{-i\alpha_0} - ie^{-i\alpha_0}A/2p. \quad (20)$$

**5. Principal vector and principal moment.** The expressions for the principal vector and principal moment of the forces applied to an arbitrary arc of the deformed contour have the form

$$P = p \int_{L_0} (\sigma_n + i\tau_n) [(1 + W'_z)e^{i\alpha_0} - W'_{\bar{z}}e^{-i\alpha_0}] ds_0, \quad (21)$$

$$M = -p \operatorname{Re} \left\{ \int_{L_0} (\sigma_n + i\tau_n)(\bar{z}' + \bar{W}')i [(1 + W'_z)e^{i\alpha_0} - W'_{\bar{z}}e^{-i\alpha_0}] ds_0 \right\}, \quad (22)$$

where the prime denotes belonging to the contour, and the integration is carried out over the undeformed contour  $L_0$ .

Equations (14), (15), (16), (20), (21), (22) and the physical laws  $\nu = f_1(e^\nu\Sigma, e^\nu T)$ ,  $\varepsilon = f_2(e^\nu\Sigma, e^\nu T)$  completely exhaust the formulation of problems of plane deformation of an ideally elastic isotropic body.

For the solution of specific problems it is recommended to use the procedure of the small parameter <sup>(3)</sup> and conformal mappings <sup>(6)</sup>.

As a result of the calculation, taking into account three approximations for the physical laws (10), an expression was obtained for the stress concentration coefficient  $\chi = \sigma_2 : p$  at the edge of a cylindrical cavity in the initial state,

$$\chi = \frac{\sigma_2}{p} = \mp 2 \left[ 1 + \frac{1}{8(1-\nu)} \frac{p}{G} + \frac{17 - 20\nu - 6g_3(1 + \nu - 2\nu^2)}{288(1-\nu)^2} \frac{p^2}{G} \right],$$

where  $\sigma_2$  is the maximum stress, and  $p$  is a uniform load at infinity.

The second term in the last formula takes into account mainly the geometric nonlinearity, and the third also the physical nonlinearity by means of the constants  $k_3, g_3$ . Analysis shows that the magnitude of the stress concentration coefficient depends on the sign of the load at infinity.

Tula Polytechnic Institute

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*Note: Figure translations are in progress. See original paper for figures.*

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