FIBERED MODULES AND COBORDISMS

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Abstract

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FIBERED MODULES AND COBORDISMS

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The paper considers questions connected with the specification of a module structure on a vector bundle, and cobordisms based on the specification of module structures in the normal bundle of a manifold. If the field of complex numbers or the skew field of quaternions is taken as the ring of operators of the module structure, then we obtain the notion of a quascomplex or quasisymplectic structure. The corresponding cobordisms are considered in detail in ([1], [4]).

Everywhere below K is the field of real or complex numbers, and Λ is a finite-dimensional associative K-algebra with unit. A fibered right Λ-module is a K-vector bundle with a prescribed right action of the algebra Λ on it by means of K-automorphisms of the vector bundle that are identical on the base. A mapping of fibered Λ-modules is a mapping of vector bundles that commutes with the action of the operators. Fibered Λ-modules and their mappings form a category.

A fibered Λ-module ξ_Λ is called linearly homogeneous if the fibers lying over points of one component of linear connectedness of the base B_ξ of the module ξ_Λ are Λ-isomorphic; the Λ-module A, to which all fibers of the fibered Λ-module ξ_Λ lying over some component of linear connectedness are isomorphic, is called a typical fiber over the given component of linear connectedness.

A fibered Λ-module is called Λ-trivial if there exists a Λ-module A such that the fibered Λ-modules ξ_Λ and $i(A) = (B_\xi \times A, \pi, B_\xi)$ are Λ-isomorphic, where B_ξ is the base for ξ_Λ and $\pi(b, a) = b$, $b \in B_\xi$, $a \in A$. ξ_Λ is called locally Λ-trivial if there exists an open covering of its base such that its restriction to any element of the covering is a Λ-trivial fibered module.

Theorem 1. *A fibered Λ-module ξ_Λ, whose base is a connected CW-complex, is locally Λ-trivial if and only if it is linearly homogeneous.*

For certain classes of algebras the question of linear homogeneity of fibered modules over them is easily resolved. For example, if Λ is a semisimple algebra or $\Lambda = K[X]/\{f(X)\}$, where $K[X]$ is the algebra of polynomials in the variable X over K, and $\{f(X)\}$ is the ideal generated by the polynomial $f(X)$, then every fibered module is linearly homogeneous. Since the indicated example of algebras is the leading one for us, in the subsequent considerations we restrict ourselves to locally Λ-trivial fibered modules, which, as follows from the assertions formulated above, is not a restriction for the indicated types of algebras. Thus, unless otherwise stated, by a fibered module we shall mean a locally Λ-trivial fibered module with base a CW-complex.

In the category of fibered Λ-modules the notions of sum, direct product of fibered modules, and also the notion of the inverse image $f!(\xi_\Lambda)$ of a fibered Λ-module ξ_Λ under a continuous mapping $f : X \to B_\xi$ of a CW-complex X into the base B_ξ of the fibered module ξ_Λ, are defined in the obvious way. As in the case of vector bundles, the following assertion holds: if

$f, g : X \to B_\xi$, and f is homotopic to g, then the fibered Λ-modules $f!(\zeta_\Lambda)$ and $g!(\zeta_\Lambda)$ are Λ-isomorphic.

Denote by $R(\Lambda, A)$ the category of fibered Λ-modules with typical fiber A. For this category there exists a universal fibered Λ-module. Namely:

Theorem 2. There exists such a fibered Λ-module $\vartheta(A)$ that for every $\zeta_\Lambda \in R(\Lambda, A)$ there exists, uniquely up to homotopy, a map $f_\xi : B_\xi \to B_{\vartheta(A)}$ such that $\zeta_\Lambda = f!\vartheta(A)$.

The proof of the last two assertions is based on the fact that assigning a module structure on the bundle ζ is equivalent to specifying a reduction of the structure group to a certain subgroup. Taking this into account, both assertions follow from analogous assertions for principal fiber spaces with arbitrary structure group.

Let, for example, $K = R$, and let $f(X)$ be a real polynomial of degree n without multiple roots, of which k roots are real and $2l$ roots are complex $(k + 2l = n)$; let A be the sum of m copies of free monogenic modules over the algebra $\Lambda = R[X]/\{f(X)\}$.

Lemma 1. The universal fibered Λ-module $\vartheta(A)$ for the category $R(\Lambda, A)$ is K-isomorphic to the bundle

$$\left(\prod_1^k \vartheta_m(R)\right) \times \left(\prod_1^l \vartheta_m(C)\right),$$

where $\vartheta_m(K)$ is the universal m-dimensional K-vector bundle, and $\Pi, \times$ denote the operation of taking the direct sum of bundles.

The case in which the polynomial $f(X)$ has multiple roots differs essentially from the preceding case. For example, if $f(X) = X^2$, then we have

Lemma 2. The universal fibered Λ-module $\vartheta(A)$, where A is the sum of m copies of free monogenic Λ-modules over the algebra $\Lambda = R[X]/\{X^2\}$, is K-isomorphic to the vector bundle $\vartheta_m(R) \oplus \vartheta_m(R)$, where $\oplus$ is the operation of taking the direct sum of bundles.

Now let $\Lambda = K(n)$ be the algebra of $n \times n$-matrices, B the canonical irreducible Λ-module (i.e. $B = K^n$, the arithmetic space on which $K(n)$ acts on the right in the canonical way–row by column), and let A be the direct sum of m copies of the module B.

Lemma 3. The universal fibered module for the category $R(\Lambda, A)$ is isomorphic to the vector bundle

$$\sum_1 \vartheta_m(K),$$

where $\sum$ denotes here the operation of taking the direct sum of bundles.

Let A be a right Λ-module; A^m the sum of m copies of the module A. The series $A, A^2, \dots, A^m, \dots$ of right Λ-modules gives rise to the series $R(\Lambda, A)$, $R(\Lambda, A^2), \dots, R(\Lambda, A^m), \dots$ of categories and to the corresponding series of universal fibered Λ-modules $\vartheta(A), \vartheta(A^2), \dots, \vartheta(A^m), \dots$. The embeddings of modules $i_m : A^m \to A^{m+1}$ induce maps $J_m : \vartheta(A^m) \oplus i(A) \to \vartheta(A^{m+1})$ of fibered modules, where $i(A)$ is a Λ-trivial fibered module with fiber A.

We shall say that a K-vector bundle ξ admits a Λ-module structure with typical fiber an isotopic module of type A, or, more briefly, a stable A-structure, if it admits a Λ-module structure after adding a trivial bundle of some dimension and the obtained fibered Λ-module belongs to one of the categories $R(\Lambda, A^m)$ for some m. A stable A-structure on a bundle ξ is the homotopy class of an action of the algebra Λ on the stable vector bundle associated with ξ.

Analogously to (1-3), one may define the notion of cobordisms of smooth manifolds with a fixed stable A-structure in the stable normal bundle of the manifold under some smooth embedding of it into a sphere of high dimension.

There arise groups $\Omega_i(A)$ of i-dimensional cobordisms of smooth manifolds with stable A-structure. Since the direct product of manifolds equipped with a stable A-structure is canonically endowed with a stable A-structure, and this is compatible with the cobordism relation,

dism, then the graded group $\Omega_*(A) = \sum_{i \geq 0} \Omega_i(A)$ is endowed with the structure of a graded commutative ring with identity.

Lemma 4.

$$\Omega_i(A) = \lim_{m \to \infty} \operatorname{ind}\{\pi_{km+i}(T\vartheta(A^m))\} = \pi_i(T\vartheta(A)), \qquad i \geq 0,$$

$m \geq 1,\ k = \dim_R A,\ T\vartheta(A) = \{T\vartheta(A^m)\}$ is a superspectrum of Thom complexes of universal stratified modules.

Moreover, since the spectrum $T\vartheta(A)$ is multiplicative, the total homotopy group of this spectrum is endowed with the structure of a graded ring and, as follows from the properties of the pairing of the spectrum $T\vartheta(A)$, this graded ring is associative, commutative, and has an identity. Let

$$\pi(T\vartheta(A)) = \sum_{i\geq 0} \pi_i(T\vartheta(A))$$

be the homotopy ring of the spectrum $T\vartheta(A)$. Then the preceding assertion about the groups $\Omega_i(A)$ admits a strengthening: the rings $\Omega_*(A)$ and $\pi(T\vartheta(A))$ are isomorphic. Taking for A a free monogenic Λ-module over the algebra $\Lambda = R[X]/\{f(X)\}$, we obtain:

$$\Omega_*(A) = \pi\left(\left\{T\left(\prod_1^k \vartheta_m(R)\right) \wedge T\left(\prod_1^l \vartheta_m(C)\right)\right\}\right) =$$

$$= \pi\left(\left\{\left(\bigwedge_1^k T\vartheta_m(R)\right) \wedge \left(\bigwedge_1^l T\vartheta_m(C)\right)\right\}\right),$$

where $\bigwedge$ is the operation of joining spectra, whose definition is given below.

If A is an irreducible $K(n)$-module, then we obtain

$$\Omega_*(A) = \pi\left(\left\{T\left(\sum_1^n \vartheta_m(K)\right)\right\}\right).$$

Below we give computations of the homotopy groups of the spectra obtained in the preceding examples. If $X = (X_n, f_n)$, $Y = (Y_n, g_n)$ are superspectra, then their join is the spectrum

$$X \bigwedge Y = (X_n \bigwedge Y_n,\ f_n \bigwedge g_n).$$

The group of X-homology of the spectrum Y is the group

$$X(Y) = \sum_{i\geq 0} X_i(Y), \qquad X_i(Y) = \lim_{n\to\infty} \operatorname{ind}(\{X_{n+i}(Y_n^{\cdot})\}).$$

Here the homology theory generated by a spectrum is denoted by the same symbol as the spectrum. If the spectra X and Y are both multiplicative, then the spectrum $X \bigwedge Y$ is also multiplicative, and therefore the group $\pi(X \bigwedge Y)$ is a graded ring. In this case the group $X(Y)$ is also a graded ring.

Theorem 3. *The rings $\pi(X \bigwedge Y)$ and $X(Y)$ are isomorphic.*

This theorem makes it possible to compute completely the homotopy rings

$$\pi\left(\left(\bigwedge_1^k T\vartheta(R)\right) \wedge \left(\bigwedge_1^l T\vartheta(C)\right)\right),$$

and hence the corresponding cobordism rings.

Let, for example, A be a free monogenic module over the algebra

$$\Lambda = R[X]/\{f(X)\}, \qquad f(X) = X^4 + 1.$$

Then

$$\Omega_*(A) = \Omega_U[x_1, x_2, \dots, x_n, \dots]$$

is the polynomial ring in the variables $x_1, x_2, \dots, x_n, \dots$, $\dim x_n = 2n$, over the ring Ω_U of U-cobordisms of a point.

If A is taken as before, $\Lambda = R[X]/\{f(X)\}$, $f(X) = X^3 + 1$, then the ring

$$\Omega_*(A) = \Omega_O[x_1, x_2, \dots, x_n, \dots],$$

where Ω_O is the ring of O-cobordisms of a point.

One can also indicate geometric generators for cobordism rings of this kind. The study of spectra of the form

$$\left\{T\left(\sum_1^n \vartheta_m(K)\right)\right\}$$

leads to the following (restricting ourselves to the case $K = C$):

$$\operatorname{rk}\left(\pi_{2i}\left(T\left(\sum_1^n \vartheta_m(C)\right)\right) \otimes Q\right) = \operatorname{rk}\left(\Omega_{2i}^U \otimes Q\right),$$

$$\operatorname{rk}\left(\pi_{2i+1}\left(T\left(\sum_1^n \vartheta_m(C)\right) \otimes Q\right)\right) = 0,$$

where Q is the field of rational numbers.

The computation of the twisted parts of these homotopy groups, however, encounters difficulties. Namely, let, for example, $X = \{T(\vartheta_m(C) \oplus \vartheta_m(C))\}$. The cohomology with coefficients in the group Z_2 of the space $T(\vartheta_m \otimes \vartheta_m)$ is mapped isomorphically onto the ideal of the ring $H^*(BU_m; Z_2)$ generated by the element C_m^2, where C_m is the top Chern class of this ring. We have the relation $Sq^4 C_m^2 = C_1^2 C_m^2$, $Sq^2(C_1 C_m^2) = C_1^2 C_m^2 = Sq^4 C_m^2$, and at the same time $C_1 C_m^2$ is not the value of any cohomology operation on the element C_m^2. This means that the cohomology of the spectrum X, regarded as a module over the Steenrod algebra, is not representable as a sum of monogenic modules, as happens in the classical case.

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Note: Figure translations are in progress. See original paper for figures.