SCATTERING THEORY AND THE “NONPHYSICAL SHEET” FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

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SCATTERING THEORY AND THE "NON-PHYSICAL SHEET" FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Let E be an n-dimensional complex Euclidean space, $n < \infty$; let $Q(x)$ be a piecewise-continuous Hermitian $n \times n$ matrix function, $-a \leq x < \infty$, $0 < a < \infty$, and moreover* $Q(x) = 0$ for $x > 0$, $Q(x) \geq 0$, $x < 0$. In the space $L_2(-a, \infty; E)$ consider the self-adjoint operator L generated by the differential expression

$$lu = -u'' + Q(x)u, \qquad x > -a,$$

and the boundary condition $u(-a) = 0$. Alongside it we consider the corresponding unperturbed operator L_0, generated in the space $L_2(0, \infty; E)$ by the differential expression $l_0 u = -u''$ and the boundary condition $u(0) = 0$.

As is known (see ([1])), the resolving operators** $V(t)$ and $V_0(t)$ for the corresponding wave equations

$$u_{tt} = -Lu, \qquad u_{tt} = -L_0 u$$

form unitary groups in the spaces $\mathcal{H}$ and $\mathcal{H}_0$ of $2n$-component vector functions (data) $U = [u_0, u_1]$, defined on $(-a, \infty)$ and $(0, \infty)$, respectively. The role of the unit form in $\mathcal{H}$ and $\mathcal{H}_0$ is played by the energy:

$$\|U\|_H^2 = \tfrac{1}{2}\{L\langle u_0, u_0\rangle + (u_1, u_1)\},$$

$$\|U\|_{H_0}^2 = \tfrac{1}{2}\{L_0\langle u_0, u_0\rangle + (u_1, u_1)_0\}.$$

Here $L\langle u_0, u_0\rangle$ and $L_0\langle u_0, u_0\rangle$ are the quadratic forms of the operators L and L_0, while (u_1, u_1), $(u_1, u_1)_0$ are the scalar products in $L_2(-a, \infty; E)$ and $L_2(0, \infty; E)$. The space $\mathcal{H}_0$ is isometrically embedded in $\mathcal{H}$ as a subspace, and the groups

$\{V(t)\}$ and $\{V_0(t)\}$ possess common incoming and outgoing subspaces D_- and D_+ (see the definition in ([1])):

$$D_- = \{U;\ U \in \mathcal{H}_0,\ u_0' = u_1\}, \qquad D_+ = \{U;\ U \in \mathcal{H}_0,\ u_0' = -u_1\}.$$

Moreover D_- and D_+ are orthogonal and together span all of $\mathcal{H}_0$.

Using the Adamyan-Arov scheme ([2]), one can define the scattering operator for the groups $\{V_0(t)\}$ and $\{V(t)\}$. The following assertion gives an explicit form for the scattering operator $S(V_0, V)$ in the spectral representation (see ([1])) of the group $\{V(t)\}$.

Let $f(x,k)$ denote the solution of the equation

$$lf(x,k) = k^2 f(x,k), \qquad f(x,k) \in E \times E, \qquad x > -a, \tag{1}$$

having the property $f(x,k)e^{ikx} = I,\ x > 0,\ \operatorname{Im} k = 0$ (see ([3])).

* The condition of nonnegativity of $Q(x)$ for $x < 0$ is not essential, but it makes it possible to simplify considerably a number of arguments in comparison with the general case (an analogous situation is discussed, for example, in ([1]), Ch. 6).

** The action of the operators $V(t)$ and $V_0(t)$ consists in taking the given solution data

$$U_0 = \left[u(0), \frac{du}{dt}(0)\right],$$

corresponding to the initial instant of time, into the solution data

$$U(t) = \left[u(t), \frac{du}{dt}(t)\right],$$

corresponding to the instant t.

Theorem 1. In the spectral representation of the group $\{V(t)\}$, the scattering operator $S(V_0, V)$ coincides with the operator of multiplication in $L_2(-\infty, \infty; E)$ by the matrix function

$$\mathcal{S}(k) = f^{-1}(-a, -k) f(-a, k), \qquad -\infty < k < \infty. \tag{2}$$

The matrix $\mathcal{S}(k)$ is called the suboperator of scattering (or scattering matrix) for the pair $^*(V_0, V)$. By virtue of the orthogonality of D_- and D_+, $\mathcal{S}(k)$ is an inner function in the upper half-plane. On the entire complex plane $\mathcal{S}(k)$ is continued as a meromorphic function with poles in the lower half-plane.

Let $\mathcal{K} = \mathcal{H} \ominus \mathcal{H}_0$, and let $P\mathcal{K}$ be the orthoprojector onto $\mathcal{K}$ in $\mathcal{H}$. The subspace $\mathcal{K}$ consists of all data $U \in \mathcal{H}$ such that $u_1(x) = 0,\ x > 0;\ u_0(x) = u_0(0),\ x > 0$.

Consider two semigroups of contractions in $\mathcal{K}$ (see ([1], [2])):

$$Z_+(t) = P\mathcal{K}V(t)P\mathcal{K}, \qquad t > 0,$$

$$Z_-(t) = P\mathcal{K}V(t)P\mathcal{K} \qquad t < 0.$$

Denote by iB_+ (iB_-) the generator of the semigroup $\{Z_+(t)\}$ ($\{Z_-(t)\}$). Obviously, $B_- = B_+^*$. The spectrum of B_+ coincides with the set of roots ** of the scattering matrix (see ([1], [4])). In view of the reality of the operator L, the condition $\mathcal{S}(k) = \mathcal{S}^*(-\bar{k})$ is satisfied; the spectrum of B_+ lies in the upper half-plane, is symmetric with respect to the imaginary axis, and consists of eigenvalues accumulating at infinity. In what follows (Theorem 3) we shall show that all sufficiently distant roots of the scattering matrix are simple. Hence it follows that the corresponding eigenvalues k_m of the operator B_+ are simple poles of its resolvent. The normalized eigenfunctions corresponding to them have the form

$$\psi_m(x) = \begin{cases} \sqrt{2\,\mathrm{Im}\,k_m} \begin{pmatrix} -\dfrac{i}{k_m} \\ 1 \end{pmatrix} f(x, k_m)\pi_m, & x \le 0, \\ \sqrt{2\,\mathrm{Im}\,k_m} \begin{pmatrix} -\dfrac{i}{k_m} \\ 0 \end{pmatrix} \pi_m, & x > 0. \end{cases}$$

Here $f(x, k_m)$ is the solution of equation (1) introduced above and $\pi \in \mathrm{Ker}\,\mathcal{S}(k_m)$, $\|\pi_m\|_E = 1$. For the eigenfunctions of the operator B_- corresponding to the eigenvalue $\bar{k}_m$, we have

$$\varphi_m(x) = \begin{cases} \sqrt{2\,\mathrm{Im}\,k_m} \begin{pmatrix} -\dfrac{i}{\bar{k}_m} \\ 1 \end{pmatrix} f(x, -\bar{k}_m)\Delta_m, & x \le 0, \\ \sqrt{2\,\mathrm{Im}\,k_m} \begin{pmatrix} -\dfrac{i}{\bar{k}_m} \\ 0 \end{pmatrix} \Delta_m, & x > 0, \end{cases}$$

where $\Delta_m \in \mathrm{Ker}\,\mathcal{S}^*(k_m)$, $\|\Delta_m\|_E = 1$. With a consistent choice of the vectors Δ_m and π_m, the systems $\{\psi_m\}$ and $\{\varphi_m\}$ become biorthogonal. *** In this case

$$(\psi_m, \varphi_m) = 2i\ \mathrm{Im}\,k_m(\mathcal{S}'(k_m)\pi_m, \Delta_m)_E.$$

In order to connect the operators B_- and B_+ with the analytic continuation of the resolvent of the operator L onto the "nonphysical sheet," we use a formula of an abstract character, relating the resolvents of the operators

[*] As follows from (2), the scattering matrix defined in (1) coincides with $\mathcal{S}^{-1}(k)$, i.e., with the scattering matrix for the pair (V, V_0).

[**] k_0 is a root of $\mathcal{S}(k)$ if $\dim\operatorname{Ker}\mathcal{S}(k_0) > 0$; k_0 is a simple root of $\mathcal{S}(k)$ if $\mathcal{S}'(k_0)\pi \neq 0,\ \pi \in \operatorname{Ker}\mathcal{S}(k_0),\ \pi \neq 0$.

[***] If $\dim\operatorname{Ker}\mathcal{S}(k_m) = 1$, then the biorthogonality condition is satisfied automatically. This case is the most important in what follows.

L, B_+ and B_- for $\operatorname{Im} k > 0$:

$$P_H \begin{pmatrix} R_{k^2}(L) & 0 \\ 0 & R_{k^2}(L) \end{pmatrix} P_H = \frac{1}{2k}\,[R_k(B_-) - R_{-k}(B_+)]. \tag{3}$$

The right-hand side of formula (3) obviously admits analytic continuation into the lower half-plane $\operatorname{Im} k < 0$ as a meromorphic function with poles at the points $\overline{k_m}$. Consequently, the left-hand side of formula (3) also admits analytic continuation into the lower half-plane, which corresponds to the analytic continuation of the resolvent $R_{k^2}(L)$ of the operator L to the "unphysical sheet." The function continued in this way

$$P_H \begin{pmatrix} R_{k^2}(L) & 0 \\ 0 & R_{k^2}(L) \end{pmatrix} P_H$$

turns out to be meromorphic in the whole k-plane, and its poles coincide with the poles of the scattering matrix. Using formula (3), one can calculate the principal part of the resolvent $R_{k^2}(L)$ at a simple pole $\overline{k_m}$ (we assume that $\dim\operatorname{Ker}\mathcal{S}(k_m) = \dim\operatorname{Ker}\mathcal{S}^*(k_m) = 1$)

$$(R_{k^2} f, g) = \frac{i}{2\overline{k_m}\,(\pi_m, S'(-\overline{k_m})\Delta_m)}\,\frac{1}{\overline{k_m} - k}\times$$

$$\times \int_{-a}^{0} (f(y, -\overline{k_m})\Delta_m, g(y))_E\, dy \cdot \int_{-a}^{0} (f(x), f(x, k_m)\pi_m)_E\, dx. \tag{4}$$

Here $f, g \in L_2(-a, 0; E)$. In the case $\dim\operatorname{Ker}\mathcal{S}(k_m) > 1$, with a consistent choice of the systems $\{\Delta_m^j\}_{j\geq 1}$, $\{\pi_m^j\}_{j\geq 1}$ in $\operatorname{Ker}\mathcal{S}^*(k_m)$ and in $\operatorname{Ker}\mathcal{S}(k_m)$, formula (4) is preserved; one need only perform summation over j, $j = 1, 2, \ldots, \dim\operatorname{Ker}\mathcal{S}(k_m)$.

2. Let us examine in more detail the analytic properties of the scattering matrix $\mathcal{S}(k)$. We shall assume that the potential $Q(x)$ is left-continuous. In this case there exists in E a left-continuous monotone family of orthoprojectors $\{p(x)\}$ such that the condition

$$Q(x')p(x) = 0, \qquad x' > x, \qquad p(+0) = I,$$

is fulfilled, and $p(x)$ is the maximal projector satisfying this condition. Denote by $-a_j$ the discontinuity points of the system $\{p(x)\}$, and let

$$p_j = p(-a_j + 0) - p(-a_j). \tag{5}$$

Everywhere in what follows we assume that

1°. The matrix-function $Q(x)p_j$ is continuous and continuously differentiable l_0 times for $x \leq -a_j$.

It is not hard to see that, under condition 1°, the subspaces p_{jE} reduce the operator

$$Q^{(s)}(-a_j) = \frac{d^sQ}{dx^s}(-a_j), \qquad s \leq l_0.$$

Let now $E'_{js} \subset p_{jE}$ be a subspace on which the matrix $Q^{(s)}(-a_j)$ is nondegenerate, $Q^{(s)}(-a_j) \times [p_{jE} \ominus E'_{js}] = 0$,

$$E_{js} = \overline{E'_{j0} + E'_{j1} + \cdots + E'_{js}},$$

and let p_{js} be the orthoprojector onto $E_{js} \ominus E_{js-1}$. We assume that the condition

2°. $E_{jl_0} = p_{jE}$ and the matrix $p_{js}Q^{(s)}(-a_j)p_{js}$ is nondegenerate if $p_{js} \neq 0$.

Next, denote by $q_{jst} \equiv q_\sigma$, $t \geq 1$, the eigenvalues of the operator $p_{js}Q^{(s)}(-a_j)p_{js}$ in the subspace $p_{js}E$, and by $p_{jst} = p_\sigma$ the orthoprojectors onto the corresponding eigenspaces. To avoid cumbersome notation, we shall agree to regard them as one-dimensional. Obviously,

$$p_{jE} = \sum_s \oplus\, p_{js}E = \sum_{st}^{*} \oplus\, p_{jst}E.$$

Theorem 2. If conditions $1^\circ, 2^\circ$ are satisfied, then the following assertions hold: a) all roots of $\mathcal{S}(k)$ lying outside a sufficiently large circle are simple; b) the set $\{k_m\}$ of roots of the scattering matrix asymptotically decomposes into n, $n = \dim E$, sequences $\{k_l^\sigma\}$ in such a way that, for the orthoprojectors P in E onto $\operatorname{Ker} \mathcal{S}(k_l^\sigma)$ and $\operatorname{Ker} \mathcal{S}^*(k_l^\sigma)$, the relations

$$P_{\operatorname{Ker} \mathcal{S}^*(k_l^\sigma)} \to p_\sigma, \quad l \to \pm\infty, \qquad P_{\operatorname{Ker} \mathcal{S}(k_l^\sigma)} \to p_\sigma, \quad l \to \pm\infty, \tag{6}$$

hold, where convergence is understood in the operator norm in E; c) correspondingly, the systems of eigenvectors $\{\psi_l^\sigma\}$ and $\{\varphi_l^\sigma\}$ of the operators B_+ and B_-, corresponding to the eigenvalues $\{k_l^\sigma\}$ and $\{\bar{k}_l^\sigma\}$, decompose into series

$$\{\psi\}^\sigma = \{\psi_l^\sigma\}_{l=\dots,-1,0,1,2\dots}, \qquad \{\varphi\}^\sigma = \{\varphi_l^\sigma\}_{l=\dots,-1,0,1,2\dots},$$

which are asymptotically orthogonal

$$(\psi_l^\sigma, \psi_l^{\sigma'}) \to 0, \quad l \to \pm\infty, \quad \sigma \neq \sigma', \qquad (\varphi_l^\sigma, \varphi_l^{\sigma'}) \to 0, \quad l \to \pm\infty, \quad \sigma \neq \sigma'.$$

d) for the eigenvalues* $k_l^\sigma = k_l^{jst}$ corresponding to the series $\{\psi\}^\sigma$, the following asymptotics** is valid as $l \to \pm\infty$ (Ln is the principal value of the logarithm):

$$k_l^{jst} = \frac{\pi l}{a - a_j} - i\,\frac{s+2}{a-a_j} \operatorname{Ln} \frac{li}{a-a_j} - i\,\frac{1}{2(a-a_j)} \operatorname{Ln} \left(\frac{1}{2i}\right)^{2+s} q_{jst} - \frac{\pi}{2(a-a_j)} + o(1). \tag{7}$$

Remark. In formula (7) one clearly sees the "serial" structure of the set of roots of the scattering matrix, and hence also of the poles of the resolvent $R_{k^2}(L)$ on the nonphysical sheet. The coarse structure of the series is determined by the set of points a_j. A finer structure is determined by the order of the lowest derivative $Q(x)$, different from zero in the space $p_{js}E$; an even finer structure is determined by the set of eigenvalues q_{jst}, $t \geq 1$, of the operator

$$p_{js} \frac{d^{s Q}}{dx^s}(a_j) p_{js}.$$

If the operator

$$p_{js} \frac{d^{s Q}}{dx^s}(a_j) p_{js}$$

has multiple eigenvalues, then the corresponding series of eigenvalues $\{k_l^\sigma\}$ are asymptotically indistinguishable.

The described classification of the eigenfunctions and eigenvalues of the operator B_+ is related to the classification proposed in (5) of the eigensubspaces of the basic operator associated with "Carleson series." In connection with what has been said, we note that in the case under consideration the series are not

Carleson. This corresponds to the fact that the system $\{\psi_l^\sigma\}$ is not a Riesz basis. Indeed, the formula

$$(\psi_l^\sigma, \varphi_l^\sigma) \sim -2i \,\mathrm{Im}\, k_l^\sigma \,(q_\sigma)^{a/(a-a_j)}(k_l^\sigma)^{-a(s+2)/(a-a_j)}, \quad l \to \pm\infty,$$

is valid, from which it follows that the system $\{\psi_l^\sigma\}$ is not even uniformly minimal.

In conclusion we note that the analytic properties of the scattering matrix formulated in Theorem 3 make it possible to investigate the completeness of the system of eigenfunctions of the operators B_+ and B_-, as well as the completeness of the corresponding "one-component system" $\{f(x, k_m)\}$. This will be done elsewhere.

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* The numbering k_l^σ is asymptotic in l.

** In the case $\dim E = 1$, analogous asymptotics are contained in $(6, 7)$. The serial structure of the asymptotics in this case is trivial—there is only one series.

Note: Figure translations are in progress. See original paper for figures.