CODINGS OF FINITE SETS AND A COMPLETENESS CRITERION UP TO CODING IN THREE-VALUED LOGIC

MATHEMATICS

1970

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Fig. 1. Lattice G_2

Figure 1: Fig. 1. Lattice G_2

Abstract

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UDC 519.95

MATHEMATICS

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CODINGS OF FINITE SETS AND A COMPLETENESS CRITERION UP TO CODING IN THREE-VALUED LOGIC

(Presented by Academician P. S. Novikov on 26 VI 1969)

A **coding** of the set $E_k = \{0, 1, \dots, k-1\}$ is any matrix

$$K = \begin{pmatrix} \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0r} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{k-1\,1} & \alpha_{k-1\,2} & \cdots & \alpha_{k-1\,r} \end{pmatrix}, \qquad \alpha_{ij} \in E_k, \tag{1}$$

containing no identical columns and no identical rows. Codings that differ only by a permutation of columns are not distinguished by us.

Using this concept and generalizing J. von Neumann' s method of double lines ([1]), in ([2]) the notions were introduced of completeness of a system of functions of k-valued logic under a coding K and of completeness up to coding (c.t.c.). Completeness criteria in ([2]) and in the present paper are formulated in terms of precomplete classes: a given system of functions is complete under the coding K (c.t.c.) if and only if it is not contained in any of the precomplete classes under the coding K (c.t.c.).

Fig. 1. Lattice G_2

In ([2]) it is proved that the number of precomplete c.t.c. classes in k-valued logic is finite for any natural $k \geq 2$, and in two-valued logic is equal to 3. In the present paper the main efforts are directed toward establishing a completeness criterion c.t.c. in three-valued logic.

1°. For any natural $k \geq 2$ the number of distinct codings of the set E_k is finite, but this number grows rapidly as k grows. Therefore recognizing completeness c.t.c. directly from the definition is inefficient. In the first part of the paper, relations between different codings are studied in order to reduce the solution of the completeness problem c.t.c. to the solution of the completeness problem for particular codings from a comparatively small collection.

Let K_1 and K_2 be codings of the set E_k. We shall say that the coding K_1 is majorized by the coding K_2 ($K_1 \preccurlyeq K_2$), if for any system of functions, completeness under the coding K_1 implies completeness under the coding K_2. We shall say that the codings K_1 and K_2 are equivalent ($K_1 \equiv K_2$), if $K_1 \preccurlyeq K_2$ and $K_2 \preccurlyeq K_1$.

Denote by G_k the set of equivalence classes of codings of E_k, partially ordered by majorization. It is easy to see that for any natural $k \geq 2$, G_k is a lower semilattice. From Theorem 1 of ([2]) it follows that G_2 is a lattice, and all semilattices G_k for $k \geq 3$ have no

at least two maximal elements. Let us refine this result. The lattice G_2 is shown in Fig. 1. To study the maximal elements of the semilattices G_k, we introduce the concept of **special codings**.

Every special coding is completely determined if a pair of objects is given: 1) a determining set of the coding (a nonempty set $U = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$ of certain pairs of elements of E_k); 2) a standard matrix (a k-row matrix M with elements—the literal values u and λ). These objects must satisfy the following requirements: a) if a pair (i, j) belongs to U, then $i < j$; b) the matrix M contains exactly k rows, contains no identical rows, identical columns, or constant columns (i.e., columns all of whose elements take identical values).

If U and M are given, then the special coding K determined by them is constructed as follows. Let $(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)$ be an arbitrary list of all pairs of the determining set (for example, in lexicographic order). Denote the number of columns of the matrix by s.

The first k columns of the coding K are constant columns (all elements of the first column are equal to 0, of the second to 1, etc.). For each $t = 1, 2, \dots, n$ (n is the number of pairs in the set U), in the columns from the $(k + (2t-2)s + 1)$-st to the $(k + (2t-1)s)$-st we copy the matrix M, everywhere replacing u by i_t, and λ by j_t; in the columns from the $(k + (2t-1)s + 1)$-st to the $(k + 2ts)$-th we copy the matrix M, everywhere replacing u by j_t, and λ by i_t. In all, K contains $k + 2ns$ columns.

For example, in 3-valued logic, from the determining set $\{(0, 1), (1, 2)\}$ and the standard matrix

$$\begin{pmatrix} u & u \\ u & \lambda \\ \lambda & u \end{pmatrix}$$

the following special coding is constructed:

$$\begin{pmatrix} 01200111122 \\ 01201101221 \\ 01210012112 \end{pmatrix}.$$

Lemma 1. Let a special coding K have determining set $U = \{(i_1, j_1), \dots, (i_n, j_n)\}$. A system A of functions of k-valued logic is complete under the coding K if and only if, by superpositions of functions of the system A and constants, one can obtain such $4n$-ary functions $f_1, \dots, f_{2n}$ for which the following hold:

$$\begin{gathered}
f_{2t-1}(i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n) = i_t, \\
f_{2t-1}(i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n, j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n) = j_t, \\
f_{2t-1}(j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n, i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n) = i_t, \\
f_{2t-1}(j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n) = j_t, \\
f_{2t}(i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n) = j_t, \\
f_{2t}(i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n, j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n) = i_t, \\
f_{2t}(j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n, i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n) = j_t, \\
f_{2t}(j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n, i_1, i_2, \dots, i_n) = i_t \\
(t = 1, 2, \dots, n).
\end{gathered}$$

Corollary 1. Special codings whose determining sets coincide are equivalent.

Corollary 2. Every precomplete, under a special coding, class of functions of k-valued logic is a class preserving a 4-ary predicate. Moreover, if the predicate is true on some quadruple *abcd*, then this predicate is also true on the quadruples *bdac*, *dcba*, *cadb*, *badc*, *acbd*, *cdab*, *dbca*.

Let the coding K have the form (1), and let K^* be a special coding with some arbitrary standard matrix (satisfying condition b)) and with determining set

$$U = \{(a, b) \mid a < b \;\&\; (\exists i, l \in E_k)(\exists j \in \{1, 2, \dots, r\}) \\ [a = a_{ij} \;\&\; b = a_{lj}]\}.$$

Lemma 2. $K \le K^*$.

On the basis of Lemmas 1 and 2, Theorems 1 and 2 are proved.

Theorem 1. *Every maximal element of the semilattice G_k consists of special encodings for which all defining sets coincide.*

Theorem 2. *Every special encoding is contained in some maximal element of the semilattice G_k.*

It follows from Theorems 1 and 2 that the number of maximal elements in G_k is equal to $2^{k(k-1)/2} - 1$.

A set S of encodings will be called universal for k-valued logic if, for any system of functions of k-valued logic, completeness up to encoding implies completeness under some encoding from the set S.

Obviously, to construct a universal set of encodings it is sufficient to take one encoding from each maximal element of the semilattice G_k. We denote such a set by S_k.

Let $R(K)$ be the number of columns in an encoding K, $R(S) = \max R(K)$ over all encodings in the set S, $L(S)$ be the number of encodings in the set S, and $R(k) = \min R(S)$ and $L(k) = \min L(S)$ over all sets of encodings universal for k-valued logic.

Theorem 3. $(k-1)(k-2)]\log_2 k[\leq R(k) \leq k + k(k-1)]\log_2 k[$.

Theorem 4. $2^{(k-1)(k-2)/2} \leq L(k) < 2^{k(k-1)/2}$.

Both upper estimates are attained simultaneously on the universal set S_k.

Theorem 5. *Let $S = \{K_1, K_2, \dots, K_l\}$ be a set of encodings universal for k-valued logic. Every precomplete with respect to t.e. class D of functions of k-valued logic can be represented in the form*

$$D = D_1 \cap D_2 \cap \dots \cap D_l,$$

where each D_i-class is precomplete under the encoding K_i $(i = 1, 2, \dots, l)$.

2°. A universal set S_3 for 3-valued logic contains 7 encodings. We shall show that S_3 is not minimal. Denote by K_1, K_2, K_3 the following encodings of the set E_3:

$$\begin{pmatrix} 0120011 \\ 0120110 \\ 0121001 \end{pmatrix}, \quad \begin{pmatrix} 0120022 \\ 0120220 \\ 0122002 \end{pmatrix}, \quad \begin{pmatrix} 0121122 \\ 0121221 \\ 0122112 \end{pmatrix}.$$

Theorem 6. *The set of encodings $\{K_1, K_2, K_3\}$ is universal for 3-valued logic.*

Theorem 7. *Every set of encodings universal for 3-valued logic contains 3 distinct encodings K', K'', K''' such that $K' \leq K_1$, $K'' \leq K_2$, $K''' \leq K_3$.*

By Corollary 2 of Lemma 1, the set of all possible 81 quadruples of elements from E_3 is divided into 21 collections, so that any predicate defining a class precomplete under the encodings K_1, K_2, K_3 takes identical values on the quadruples of one collection.

Define the classes $A_1, A_2, \dots, A_{30}$ of functions of 3-valued logic as the preservation classes of 4-place predicates $P_1, P_2, \dots, P_{30}$. Table 1 gives the truth values of these predicates on 21 distinct quadruples (representatives of all 21 collections of quadruples). These predicates are thereby completely determined.

Theorem 8. *The classes $A_1, A_2, \dots, A_{30}$ and only they are precomplete under the encoding K_1.*

Define the classes B_i and C_i $(i = 1, 2, \dots, 30)$ as the classes of functions dual ([3]) to the classes A_i with respect to the permutations $s_B(0) = 2$, $s_B(1) = 0$, $s_B(2) = 1$ and $s_C(0) = 1$, $s_C(1) = 2$, $s_C(2) = 0$.

Theorem 8. *The classes $B_1, B_2, \dots, B_{30}$ and only they are precomplete under the encoding K_2. The classes $C_1, C_2, \dots, C_{30}$ and only they are precomplete under the encoding K_3.*

Table 1

	0000	1111	2222	0011	0022	1122	0001	0111	0110	0002	0222	0220	1112	1222	1221	0012	0120	1102	1021	2201	2012
1	u	u	u	u	u			u			u										
2	u	u	u	u	u		u			u											
3	u	u	u	u		u		u					u								
4	u	u	u	u		u	u							u							
5	u	u	u	u	u	u		u			u			u						u	
6	u	u	u	u	u	u	u			u			u			u					
7	u	u	u	u	u	u		u		u			u					u			
8	u	u	u	u	u	u	u				u			u		u					
9	u	u	u	u	u	u		u			u		u					u			
10	u	u	u	u	u	u	u			u				u						u	
11	u	u	u	u	u	u		u		u	u	u	u					u			
12	u	u	u	u	u	u	u			u	u	u		u		u	u	u		u	u
13	u	u	u	u	u	u			u	u	u	u			u			u	u		
14	u	u	u	u	u	u		u			u		u	u	u	u		u	u	u	u
15	u	u	u	u	u	u	u			u			u	u	u	u					
16	u	u	u	u	u	u			u			u	u	u	u	u	u				
17	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u
18	u	u	u	u	u	u		u	u	u	u	u	u	u	u	u	u	u	u	u	u
19	u	u	u	u	u	u	u			u	u	u	u	u	u	u	u	u		u	u
20	u	u	u	u	u	u	u			u	u	u	u	u	u	u	u	u	u	u	u
21	u	u	u	u	u	u		u		u	u	u	u	u	u	u		u	u	u	u
22	u	u	u	u	u	u		u		u	u	u	u	u	u	u	u	u	u	u	u
23	u	u	u	u	u	u			u	u	u	u	u	u	u	u	u	u	u	u	u
24	u	u	u	u	u	u				u	u	u	u	u	u	u	u	u	u	u	u
25	u	u	u	u	u	u				u	u	u	u	u	u	u		u		u	u
26	u	u	u	u				u													
27	u	u	u	u			u														
28	u	u	u	u					u												
29	u	u	u	u	u	u			u			u			u						
30	u	u	u	u	u	u											u		u		u

Theorem 9. There are exactly 65 precomplete, up to c.e., classes of functions of three-valued logic:

$$A_1,\ B_1,\ C_1,\ A_i\ (i = 5 \div 9),\ A_{29},\ A_{30},\ A_2 \cap C_l\ (l = 17 \div 25),$$
$$B_2 \cap A_l\ (l = 17 \div 25),\quad C_2 \cap B_l\ (l = 17 \div 25),\quad A_m \cap B_n\ (m = 11 \div 13,\ n = 26 \div 28),$$
$$B_m \cap C_n\ (m = 11 \div 13,\ n = 26 \div 28),\quad C_m \cap A_n\ (m = 11 \div 13,\ n = 26 \div 28).$$

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Received
26 VI 1969

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Note: Figure translations are in progress. See original paper for figures.