ON CONTINUOUS SELECTORS IN UNIFORM SPACES

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Abstract

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V. A. GEILER

ON CONTINUOUS SELECTORS IN UNIFORM SPACES

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The present note is devoted to a generalization of some theorems of E. Michael on continuous selectors of multivalued mappings ([1,2]) to the case where the multivalued mapping takes its values in a uniform space.

In what follows E will denote a certain separated uniform space, $\mathcal{U}$ the filter of entourages of its uniformity, $\mathcal{U}_0$ a certain filter of entourages of a weaker uniformity on E than the original one, having a countable base, and E_0 the uniform space obtained by endowing E with the uniformity whose filter of entourages is $\mathcal{U}_0$. By $\mathfrak{P}(E)$, $C(E)$, $B(E)$ we shall denote, respectively, the set of all subsets, all complete subsets, and all bicompact subsets of E.

Definition 1. A set $\mathfrak{A} \subset \mathfrak{P}(E)$ is called uniformly metrizable by means of $\mathcal{U}_0$ if the condition

$$(*)\ \forall V \in \mathcal{U}\ \exists U \subset \mathcal{U}_0\ \forall A \in \mathfrak{A}:\ (A \times A) \cap U \subset V.$$

is satisfied.

If E is metrizable, then, obviously, every $\mathfrak{A} \subset \mathfrak{P}(E)$ is uniformly metrizable by means of $\mathcal{U}$. If E is a topological group endowed, say, with the right uniformity, and H is a metrizable subgroup of E, then the set of right cosets modulo H is uniformly metrizable.

Theorem 1. Let X and Y be topological spaces, and let $\mathfrak{A}$ be a subset of $C(E)$ uniformly metrizable by means of $\mathcal{U}_0$. Suppose a commutative diagram is given

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathfrak{A} \\ h\downarrow & & \downarrow i \\ Y & \xrightarrow{\psi} & \mathfrak{P}(E_0) \end{array}$$

where h is continuous, φ and ψ are lower semicontinuous, and i is the inclusion. Then, if Y is perfectly normal*, there exist continuous mappings $f : X \to E$

and $g : Y \to E_0$ such that $f(x) \in \varphi(x)$, $g(y) \in \psi(y)$ $(x \in X,\ y \in Y)$, and the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ h \downarrow & & \downarrow 1 \\ Y & \xrightarrow{g} & E_0 \end{array}$$

is commutative, where 1 is the identity mapping of E onto E_0.

Proof. Let $(U_n)_{n\geq 1}$ be a base of the filter $\mathcal{U}_0$ such that

$$U_n^{-1} = U_n, \quad U_{n+1} \subset U_n$$

and all U_n are open. We construct by induction sequences $(\varphi_n)_{n\geq 0}$ and $(\psi_n)_{n\geq 0}$ of lower semicontinuous mappings

$$\varphi_n : X \to \mathfrak{P}(E), \qquad \psi_n : Y \to \mathfrak{P}(E_0)$$

such that 1) $i \circ \varphi_n = \psi_n \circ h$, 2) $\psi_{n+1}(y) \subset$

* I.e., into every open cover of Y one can inscribe an open cover consisting of pairwise disjoint sets.

$\subset \psi_n(y)$, 3) $\psi_n(y)$ is of order of smallness U_n. Put $\varphi_0 = \varphi$ and $\psi_0 = \psi$. Suppose that for $k \leq n$ the φ_k and ψ_k have been constructed. For each $a \in E$, let

$$G(a) = \{y \in Y : \psi_n(y) \cap U_{n+1}(a) \neq \emptyset\}.$$

Then $(G(a))_{a\in E}$ is an open cover of Y. Take an open refinement $(H_\lambda)_{\lambda\in L}$, inscribed in $(G(a))$, and for each $\lambda \in L$ choose $a_\lambda \in E$ so that $H_\lambda \subset G(a_\lambda)$. Let g_n be the continuous mapping of Y into E defined by the relation $g_n(y) = a_\lambda$ if $y \in H_\lambda$, and let $f_n = g_n \circ h$. Then one may put

$$\varphi_{n+1}(x) = \varphi_n(x) \cap U_{n+1}(f_n(x)), \qquad \psi_{n+1}(y) = \psi_n(y) \cap U_{n+1}(g_n(y)).$$

By virtue of 3), $(\varphi_n(x))_{n\geq 1}$ and $(\psi_n(y))_{n\geq 1}$ are bases of Cauchy filters in $\varphi(x)$ and $\psi(y)$, respectively; let $f(x)$ and $g(x)$ be their limits. We shall show that f and g are continuous. Consider, for example, f (the proof for g is analogous). Let $x_0 \in X$, $V \in \mathcal{U}$, and let $V(f(x_0))$ be the corresponding neighborhood of $f(x_0)$. Take an open $W \in \mathcal{U}$ such that

$$W^{-1} = W, \qquad \overset{2}{W} \subset V.$$

By virtue of $(*)$, there is an n such that

$$(\varphi(x) \times \varphi(x)) \cap \overset{3}{U}_n \subset W$$

for all $x \in X$. Let

$$O = \{x \in X : \varphi_n(x) \cap W(f(x_0)) \neq \emptyset\};$$

then O is open. We shall show that $x \in O$ implies $f(x) \in V(f(x_0))$. Choose

$$a_0 \in \varphi_n(x) \cap W(f(x_0));$$

by virtue of 3), $f(x) \in \overset{2}{U}_n(a_0)$. Hence $f(x) \in W(a_0)$, and therefore

$$f(x) \in W(f(x_0)) \subset V(f(x_0)).$$

The theorem is proved.

Remark. If E is a locally convex space and the sets from $\mathfrak{A}$ are convex, then it is enough to assume that Y is paracompact.

Corollary (P. Kenderov ([3])). *Let E be a locally convex space, and let F be its Fréchet subspace. Then the natural mapping $p : E \to E/F$ has a continuous right inverse.*

2. Below we shall give the definition of an almost metrizable uniform space, which is a generalization of the definition of an almost metrizable group introduced by B. A. Pasynkov ([4]).

Below, R will denote the equivalence relation in E whose graph is the set

$$\bigcap\{U : U \in \mathcal{U}_0\}.$$

Definition 2^***.** We shall call E almost metrizable with respect to $\mathcal{U}_0$ if: 1) for every $a \in E$, $R(a)$ is a bicompactum; 2) for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}_0$ such that

$$U \subset V \circ R.$$

Definition 3. A set $\mathfrak{A} \subset \mathcal{P}(E)$ is called uniformly almost metrizable with respect to $\mathcal{U}_0$ if: 1) for every $A \in \mathfrak{A}$ and every $a \in A$, $R(a) \cap A$ is a bicompactum; 2) for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}_0$ such that, for every $A \in \mathfrak{A}$,

$$(A \times A) \cap U \subset V \circ R.$$

For example, if E is a topological group endowed with the right uniformity, and H is an almost metrizable subgroup of E, then the set of right cosets modulo H is uniformly almost metrizable.

Theorem 2. *Let X and Y be topological spaces, let $\mathfrak{A}$ be a subset of $C(E)$ uniformly almost metrizable with respect to $\mathcal{U}_0$, and let the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathfrak{A} \\ h\downarrow & & \downarrow i \\ Y & \xrightarrow{\psi} & \mathcal{P}(E_0) \end{array}$$

be commutative, where h, φ, ψ, i are as in Theorem 1. If Y is perfectly zero-dimensional, then there exist lower semicontinuous mappings $\eta : X \to B(E)$ and $\theta : Y \to B(E_0)$ such that $\eta(x) \subset \varphi(x)$, $\theta(y) \subset \psi(y)$ $(x \in X,\ y \in Y)$, and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & B(E) \\ h\downarrow & & \downarrow j \\ Y & \xrightarrow{\theta} & B(E_0) \end{array}$$

is commutative,

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* This definition was proposed by P. Kenderov.

where j is an embedding. Here $\eta(x) = R(f(x)) \cap \varphi(x)$, and $\theta(x) = R(g(x)) \cap \psi(x)$, where $f : X \to E$, $g : Y \to E$ are certain single-valued mappings.

Corollary 1. Suppose that in the notation of Theorem 2 X is perfectly zero-dimensional, E is a topological group, and $\mathfrak{A}$ consists of all left cosets with respect to some almost metrizable Weil-complete subgroup $H \subset E$. Then φ has a single-valued continuous selector.

Corollary 2. Let S be an open equivalence relation in E, and let $p : E \to E/S$ be the natural mapping. If the set of cosets with respect to S is uniformly almost metrizable and is contained in $C(E)$, and the equivalence relation $R \cap S$ is closed, then for every paracompact subset $X \subset E/S$ (where E/S is endowed with the quotient topology) there exists a paracompact subset $Y \subset E$ such that $p(Y) = X$ and $p|_Y$ is perfect.

Indeed, let $\varphi : X \to \mathfrak{P}(E)$ be given by the relation $\varphi(x) = p^{-1}(x)$; then φ is lower semicontinuous. By a known result of V. I. Ponomarev ([5]), there exists a perfectly zero-dimensional space $\dot{X}$ (an absolution of X) and a perfect mapping π of $\dot{X}$ onto X. The mapping $\varphi \circ \pi$ is lower semicontinuous; therefore, by Theorem 2, there is a single-valued mapping $f : \dot{X} \to E$ such that the mapping $\theta : x \to R(f(x)) \cap S(f(x))$ is lower semicontinuous. But since $R \cap S$ is closed, θ is upper semicontinuous; consequently, one may put $Y = \theta(X)$.

In exactly the same way, from Corollary 1 one can obtain Theorem 2 of B. A. Pasynkov' s paper ([6]).

Corollary 3. Let S be an equivalence relation, open and regular* both in E and in E_0. If the set of cosets with respect to S is uniformly almost metrizable by means of the $\mathfrak{U}_0$-subset $C(E)$, and the equivalence relation $R \cap S$ is closed in E, then there exists an $X \subset E$ such that the restriction to X of the natural mapping $p : E \to E/S$ is perfect, and $p(X) = E/S$.

From Corollary 3 there follows the following theorem of P. Kenderov ([8]):

Let G be a topological group, H a Weil-complete, almost metrizable, normal subgroup of G, and $p : G \to G/H$ the natural mapping. Then there exists $X \subset G$ such that $p(X) = G$ and $p|_X$ is perfect.

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Mordovian
State University
Saransk

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* V. L. Levin and D. A. Raikov call an equivalence relation S in E regular if $\forall V \in \mathfrak{U}\ \exists U \subset \mathfrak{U} :\ S \circ U \subset V \circ S$.

Note: Figure translations are in progress. See original paper for figures.