TRANSFORMATION PROPERTIES OF GENERALLY COVARIANT OBSERVABLES

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Abstract

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TRANSFORMATION PROPERTIES OF GENERALLY COVARIANT OBSERVABLES

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When Cartesian coordinates are used in the traditional formulation of special relativity, finding the physically observable quantities presents no problem. But, as V. A. Fock showed with particular clarity (1), special theory can be set forth in arbitrary coordinates, like general relativity (G.R.), differing from it only by the zero curvature of space-time. In this case, and especially in G.R., the definition and analysis of the properties of observables form an independent problem.

Without touching here upon the tetrad approach to observables, we shall choose the mathematically more economical concept of A. L. Zelmanov (2,3) (the formalism of chronometric invariants), in which a reference frame can be realized by a reference body, the points R of which are at rest in the given coordinate system, so that their world lines coincide with the lines of coordinate time. Later Zelmanov gave a generalization of his formalism, free of the need to relate a reference frame to one or another coordinate system (4).* The quantities appearing in the generally covariant approach he called "assigned quantities," emphasizing their connection with observables in the given reference frame. The congruence of world lines of the reference body determines the field of a unit timelike vector

$$\tau^\mu = (dx^\mu/ds)_R \tag{1}$$

and the field of the "spatial metric" tensor

$$b_{\mu\nu} = \tau_\mu \tau_\nu - g_{\mu\nu} \tag{2}$$

(as is easy to see, $\det b_{\mu\nu} \equiv 0$). The vector τ^μ and the tensor $b_{\mu\nu}$ make it possible to define, in a covariant manner, the assigned quantities for the given reference frame,** since τ^μ carries out the projection of tensor quantities onto the direction of physical time (the proper time of the given reference frame), while $b_{\mu\nu}$ projects them onto the spacelike 3-manifold orthogonal to it (4), since $\tau^\mu b_{\mu\nu} \equiv 0$, and $b_{\mu\nu} b^{\nu\lambda} = -b_\mu{}^\lambda$, if the signature of the metric is (-2). Denoting ordinary tensor quantities by lowercase letters and the corresponding assigned quantities by uppercase letters, we define

$$P_* = p^\mu \tau_\mu, \qquad P_\mu = p^\nu b_{\mu\nu}, \tag{3}$$

so that, as in the usual chronometrically invariant notation,

$$a_\mu c^\mu \equiv A_* C_* - (AC); \qquad (AC) \overset{\text{Def}}{=} b^{\mu\nu} A_\mu C_\nu . \tag{4}$$

The generalization to the case of a tensor of arbitrary rank is obvious; for example, from $t_{\mu\nu}$ we obtain T_{**}, $T_{*\nu}$, $T_{\mu *}$, and $T_{\mu\nu}$.

* In particular, this is important for coordinate systems in which $g_{00} = 0$, which in general fell outside the scope of the usual formalism of chronometric invariants.

** This approach was first proposed by Eckart (^{5}).

Our task is to find the transformation laws of "observables" of type (3) under transitions between different reference systems; therefore we do not need the differential relations of Zel' manov' s new formalism. In contrast to paper (5), we can now use a single coordinate system, specifying in it the two reference systems being compared by the vectors τ^μ and $\bar\tau^\mu$. In the system τ^μ, a 4-displacement dx^μ of each point $\bar R$ at rest in the system $\bar\tau^\mu$ corresponds to spatial and temporal displacements $dL_\mu = b_{\mu\nu} dx^\nu_{\bar R}$ and $dT = \tau_\mu dx^\mu_{\bar R}$, with $ds^2_{\bar R} = dT^2 - dL^2$ in accordance with (4). Then the velocity of the points $\bar R$ in the system τ^μ (i.e., the field of relative velocity of the reference systems from the viewpoint of the system τ^μ) is defined as $V_\mu = (dL_\mu/dT)_{\bar R}$. Similarly, from the viewpoint of the system $\bar\tau^\mu$, for points R at rest in the system τ^μ, we have $\bar V_\mu = (d\bar L_\mu/d\bar T)_R$. Since $\bar\tau^\nu b_{\mu\nu} = (dL_\mu/ds)_{\bar R}$ and $\tau^\alpha \bar\tau_\alpha = (dT/ds)_{\bar R}$, we obtain the explicit expression for V_μ in terms of $\tau^\mu, \bar\tau^\mu$, and $b_{\mu\nu}$:

$$V_\mu = \frac{\bar\tau^\nu b_{\mu\nu}}{\tau^\alpha \bar\tau_\alpha}, \qquad \text{analogously,} \qquad \bar V_\mu = \frac{\tau^\nu \bar b_{\mu\nu}}{\tau^\alpha \bar\tau_\alpha}. \tag{5}$$

It follows that $\tau^\alpha \bar\tau_\alpha = (1 - V^2)^{-1/2}$ and

$$\bar\tau_\mu = \frac{\tau_\mu - V_\mu}{\sqrt{1 - V^2}}, \tag{6}$$

$V^2 = b_{\mu\nu} V^\mu V^\nu$. From the obvious identity $\bar b_{\mu\nu} = b_{\mu\nu} - \tau_\mu \tau_\nu + \bar\tau_\mu \bar\tau_\nu$, and from (6), there follows the transformation law

$$\bar b_{\mu\nu} = b_{\mu\nu} + \frac{\tau_\mu \tau_\nu V^2 + V_\mu V_\nu - \tau_\mu V_\nu - \tau_\nu V_\mu}{1 - V^2}. \tag{7}$$

It is easy to see that

$$V^2 = \bar V^{\,2}, \qquad \bar V^{\,2} = \bar b_{\mu\nu}\bar V^{\mu}\bar V^{\nu}. \tag{8}$$

Completing the comparison of the velocities V_μ and $\bar V_\mu$, we cite the following formulas from (5):

$$\bar V_\mu \pm V_\mu = (\bar\tau_\mu \pm \tau_\mu)(1 \mp 1/\tau^\alpha \bar\tau_\alpha)$$

and

$$\frac{V^2}{\sqrt{1-\bar V^{\,2}}} = g^{\mu\nu}V_\mu V_\nu = -(V\bar V),$$

from which it is clear that the velocities $\bar V_\mu$ and V_μ are not collinear in the four-dimensional sense.

Using relation (6) in the definition $\bar P_* = p^\mu \bar\tau_\mu$, we find the transformation law of the assigned scalar

$$\bar P_* = \frac{P_* - (VP)}{\sqrt{1-V^2}}. \tag{9}$$

This law is valid for arbitrary relative motion of reference systems and, in its form, is a direct generalization of the well-known Lorentz formula. The transformation law of the assigned vector has a more cumbersome form:

$$\bar P_\mu = P_\mu + (1-V^2)^{-1} \cdot \left[P_*(\tau_\mu V^2 - V_\mu) - (VP)\cdot(\tau_\mu - V_\mu)\right] \tag{10}$$

(it follows from definition (3) and law (7)). Contracting (10) with p^μ and assuming $P_\mu \parallel V_\mu$, we obtain the Lorentz-form transformation law of the modulus:

$$\bar P = \frac{P - VP_*}{\sqrt{1-V^2}}. \tag{11}$$

We also give the transformation law for the scalar product of assigned vectors:

$$\overline{(AC)} = (AC) - A_*C_* + \frac{(A_* - (VA))(C_* - (VC))}{1-V^2}, \tag{12}$$

This law is, in form, quite similar to the one known from the traditional special theory of relativity.

Passing to the theorem of addition of physical velocities for an arbitrarily specified relative motion of reference frames, let us consider the motion of a certain point P. Its velocity in the systems τ^μ and $\bar{\tau}^\mu$, respectively, is defined as

$$U_\mu = \frac{b_{\mu\nu} dx_P^\nu}{\tau_\alpha dx_P^\alpha} \quad \text{and} \quad \bar{U}_\mu = \frac{\bar{b}_{\mu\nu} dx_P^\nu}{\bar{\tau}_\alpha dx_P^\alpha}. \tag{13}$$

Substituting in $\bar{U}_\mu$ formulas (6) and (7) and expressing the corresponding terms through U_μ, we obtain the desired law for the components of velocity:

$$\bar{U}_\mu = \frac{\sqrt{1-V^2}}{1-(UV)} \left(U_\mu - \tau_\mu + \frac{(\tau_\mu - V_\mu)(1-(UV))}{1-V^2} \right). \tag{14}$$

Its form is not similar to the usual notation of the velocity-addition theorem in the special theory of relativity, but it is its generalization. Indeed, the relation

$$1 - \bar{U}^2 = \frac{(1-V^2)(1-U^2)}{(1-(UV))^2}, \tag{15}$$

holds; it follows from (13) and coincides with formula (16.11) in (1). From (15) we obtain

$$\bar{U}^2 = \frac{(UV)^2 - U^2V^2 + (U-V)^2}{(1-(UV))^2}, \tag{16}$$

whence in the case $U_\mu \parallel V_\mu$ the well-known formula follows

$$\bar{U} = \frac{U-V}{1-UV}. \tag{17}$$

Thus, we see that the transformation laws of observables in GRT (and for arbitrarily moving reference frames in the special theory of relativity) are a natural generalization of the Lorentz formulas. We have given transformations only of assigned quantities constructed from tensors of rank 1; for tensors of higher rank they can be derived by the same method. Let us note that the right-hand sides of such formulas contain combinations of all assigned quantities obtained from the given tensor. For example, in the case $T_{**} = t_{\mu\nu}\tau^\mu\tau^\nu$,

$$\bar{T}_{**} = (1-V^2)^{-1} \cdot \left(T_{**} - T_{*\mu} V_\nu b^{\mu\nu} - T_{\mu *} V_\nu b^{\mu\nu} + T_{\mu\nu} V^\mu V^\nu \right). \tag{18}$$

In conclusion let us note that, in order to compare the covariant laws obtained here with the results of work (5), where transformations of the usual chronometrically invariant quantities were considered, it is necessary to pass to the coordinate system in which the world lines of the points of the reference body are lines of coordinate time, and where, consequently, we have $(\tau^\mu = (dx^\mu/ds)_R = \delta^\mu_0/\sqrt{g_{00}}$. Then from (2) it follows that $b_{00} = b_{0i} = 0$;

$$b_{ij} = \frac{g_{0i} g_{j0}}{g_{00}} - g_{ij}.$$

Moreover, the left-hand sides of the covariant laws must be transformed to the coordinate system in which $\bar{\tau}'^\mu = \delta^\mu_0/\sqrt{g'_{00}}$ (a transformation "additional" to Zelmanov' s chronometric and 3-dimensional transformations). Then, for example, from the transformation law (10) there follows the transformation law of a chronometrically invariant vector under passage to a new reference frame and simultaneous passage to the coordinate system associated with it, in the form known from (5):

$$\bar{P}'^k = \frac{\partial x'^k}{\partial x^i}\left[P^i - P_* V^i \frac{\left(1 - \frac{g_{0m}}{\sqrt{g_{00}}} P^m/P_*\right)}{\left(1 - \frac{g_{0n}}{\sqrt{g_{00}}} V^n\right)}\right]. \qquad (19)$$

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