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Abstract

Full Text

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ON THE STRUCTURE OF THE COEFFICIENT REGION OF UNIVALENT FUNCTIONS OF CLASS S

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1.

We shall use the notation and definitions introduced in our notes (^{8,9}). Below we shall study the differential properties of the boundary of the coefficient region $\mathcal{V}_n$. If $x \in \mathcal{A}$, then $\dim \Lambda_x > 1$. Hence it follows that the extremal function $f(z) = f(z, x)$ satisfies at least two equations

$$Q_1(f)(Df/f)^2 = -P_1(z), \qquad Q_2(f)(Df/f)^2 = -P_2(z), \tag{1}$$

where $Q_1(w)$ and $Q_2(w)$ are not proportional to one another, and of the functions P_1, P_2 at least one, say P_1, satisfies the condition $P_1(z) \geq 0$ for $|z| = 1$. Schaeffer and Spencer, in their monograph (^{1}), considered univalent functions of this kind and showed that they are algebraic (this is obtained by dividing one equation by the other) and that for simple n from equations (1) we obtain $f(z) = z(1 - \eta z)^{-2}$, $|\eta| = 1$. It can be shown that $f(z, x)$, $x \in \mathcal{A}$, satisfies an equation of the form

$$q[f] - p(z) = 0, \tag{2}$$

where $q(w)$, $q(\infty) = 0$, is a polynomial in $1/w$ of degree r, and

$$p(z) = \sum_{-r}^{r} b_k z^k, \qquad \operatorname{Im} p(z) = 0, \quad |z| = 1.$$

If at least one of the functions Q_j, $j = 1, 2$, is a polynomial in $1/w$ of exact degree n, then $r \mid n$. From (2) we obtain that, for $|z| = 1$, $\operatorname{Im} q[f] = 0$, i.e. the tree $\mathcal{T}$ is contained in the graph of the zero-level lines of the harmonic function $\operatorname{Im} q(w)$. From (2) it is easy to obtain differential equations of the form (1) satisfied by the function f. It may happen that there exists a constant α such that $q(w) + \alpha$ and $p(z) + \alpha$ have only multiple zeros (if $\alpha = 0$, a simple zero $w = \infty$ is allowed). Then the differential equation satisfied by f can be written in the form

$$\frac{[Dq(f)]^2}{q(f)+\alpha}\sum_{l=0}^{s}\gamma_l[q(f)]^l\left(\frac{Df}{f}\right)^2-\frac{(Dp(z))^2}{p(z)+\alpha}\sum_{l=0}^{s}\gamma_l[p(z)]^l=0, \tag{3}$$

where γ_l are real numbers. If such a constant α does not exist, then the differential equation is written in the form

$$[Dq(f)]^2\sum_{l=0}^{s}\gamma_l[q(f)]^l\left(\frac{Df}{f}\right)^2-(Dp(z))^2\sum_{l=0}^{s}\gamma_l[p(z)]^l=0. \tag{4}$$

It is clear that the set of equations satisfied by f is described by formulas (3) or (4) with arbitrary real constants γ_l. It is easy to describe the set of univalent extremal functions $f(z,x)$ such that $x\in\mathcal{A}$. Let $q(w)$, $q(\infty)=0$, be an arbitrary polynomial in $1/w$ of degree r. Take the graph of the zero-level lines of the function $\operatorname{Im} q(w)$ and choose from this graph a tree $\mathcal{T}$ so that it has its root at the point $w=\infty$ and exterior conformal radius 1. A function of class S,

mapping the disk $|z|<1$ onto the complement of this tree will satisfy equation (2) with the corresponding quasipolynomial $p(z)$. And then, as a consequence, it will satisfy differential equations either of the form (3) or of the form (4). Using the results obtained, one can estimate the dimension of the set $\mathcal{A}$. Put

$$p=\max\{[n/2]+[n/4]+1,3[n/3]\};$$

it turns out that

$$\dim\mathcal{A}\leqslant p-1. \tag{5}$$

The set $\mathcal{A}$ is the set of singularities of the boundary $\partial\mathcal{V}_n$. If $x\in\mathcal{A}$, then at the point x, generally speaking, there is no tangent hyperplane. The cone $K_e(x)$ has a simple geometric meaning, since any vector $\lambda\in K_e(x)$ is the limit of a sequence of vectors $\{\lambda^k\}$, $\lambda^k\in K_e(x^k)$, $x^k\in R$, $k=1,\ldots$

2. It was noted above that the degree of smoothness of the boundary $\mathcal{V}_n$ in a neighborhood of a point $x\in R$ depends on the structure of the boundary tree $\mathcal{T}$ of the function $f(z;x)$. If $Q(w)(dw/w)^2$ is the corresponding quadratic differential, then any non-terminal vertex w_j of the tree $\mathcal{T}$ will be a zero of the function $Q(w)$. We denote the multiplicity of this zero by $\varkappa_j$ and call this number the index of the vertex. To each vertex w_j of the tree $\mathcal{T}$ we assign two integers: its index $\varkappa_j$ and its degree as a vertex of the tree. We shall allow values of the index $\varkappa_j=0$. The trees $\mathcal{T}$ will be divided into equivalence classes, and first of all we shall consider those classes of trees for which the index of any non-terminal vertex is equal to 1, the degree is equal to 3, the index of any terminal vertex is zero, and the index of the root is equal to 1. Trees of such classes will differ from one another in the number of edges. To each such class of trees on $R\subset\partial\mathcal{V}_n$ there corresponds an open set of points. It turns out that in a

neighborhood of any point of the corresponding set the boundary $\partial\mathcal{V}_n$ is an analytic hypersurface. To the set of trees not satisfying the conditions listed above there corresponds on $\partial\mathcal{V}_n$ a set of dimension less than $2n-1$.

Of especially great importance is the class of trees consisting of one edge and subject to the additional conditions that the terminal vertex of the tree is not a zero of $Q(w)$, while the root is a zero of first order. To this class there corresponds a certain open subset $R_1 \subset R$. The set R_1 is linearly connected, and its closure contains the support subset of the set $\partial\mathcal{V}_n$, i.e., the subset of those points $x \in \partial\mathcal{V}_n$ at which there exists a support hyperplane to $\mathcal{V}_n$. We note that to a support point x there corresponds a function $f(z,x)$ giving an absolute extremum to some linear functional. The theorem on the support set follows from a generalization of the well-known Schiffer lemma, given by Schaeffer and Spencer in (1), and from our results on the structure of the boundary tree of a function giving a local maximum to a linear functional.

3. The results obtained in the preceding note (^{9}) make it possible to solve the question of sufficient conditions for a local extremum of the functional $J(f) = F(a_1, \dots, a_n)$. Let the function $f(z,x)$, at which the functional is stationary, correspond to the point $x \in R$. In order that it give a local maximum to the functional, it is sufficient that

$$\max_{|t|=1,\ \mathrm{Re}\sum_1^n \bar{\lambda}_k t_k = 0} \ \sup_{\xi \in \mathfrak{Z}_0} \delta^2 F < 0 \tag{6}$$

(see the notation in (3)), provided only that the Hilbert spaces $\mathcal{H}$ and $\mathfrak{H}$ coincide. We note that the necessary condition differs from (6) only in that there will be no strict inequality. If, however, $\mathcal{H}$ is a proper subspace of $\mathfrak{H}$, it may turn out that condition (6) is no longer sufficient, and in it the upper bound must be taken not over the set $\mathfrak{Z}_0$, but over some subset of the space $\mathfrak{H}$. If $f(z,x)$ corresponds to a point $x \in \mathcal{A}$, then ne-

additional considerations are necessary. Here a typical example is the functional $F = \mathrm{Re}\, a_n$, which is stationary on the Koebe function. This case can be analyzed completely and the sufficiency for a local maximum of a correspondingly modified condition (6) can be proved. In the works of Garabedian, Ross, and Schiffer (^{2}), and also of Garabedian and Schiffer (^{3}), the local Bieberbach conjecture is investigated by means of substantially different methods.

4. In order to succeed in solving a number of concrete extremal problems, it is necessary to settle the question of the possible number of critical values of the functional $F(a_1, \dots, a_n)$. This question is difficult, and whether the number of critical values is finite or infinite is far from clear. The following considerations may shed some light on this question.

Let $l < n$, and denote by $\xi = (\xi_1, \dots, \xi_l)$ a point of the set $\mathcal{V}_l$. To each point ξ there corresponds in R^{2n} the plane $a_k = \xi_k$, $1 \le k \le l$. Denote by $T_\xi^{l,n}$ the section of $\mathcal{V}_n$ by the indicated plane. These sections possess a number of interesting properties. By $\mathrm{Intv}\, T_\xi^{l,n}$ we shall denote the set of relatively interior points of the section $T_\xi^{l,n}$. It turns out that the set $\operatorname{Intv} T_\xi^{l,n}$ is linearly connected, and always

$$\operatorname{Intv} T_\xi^{l,n} \subset \operatorname{Int} \mathcal{V}_n.$$

The set $T_\xi^{l,n}$ itself coincides with the closure of the subset of relatively interior points.

From Teichmüller' s well-known theorem it follows that the set $T_\xi^{n-1,n}$ is strictly convex. Such strict convexity will hold for $l \geq [n/2]$. Below we shall assume that $l = [n/2]$. Let $x^0 \in \partial T_\xi^{l,n} \subset \partial \mathcal{V}_n$. Any vector $\lambda = (\lambda_1, \dots, \lambda_n) \in K_e(x^0)$ determines a supporting normal $\lambda^0 = (\lambda_{l+1}, \dots, \lambda_n)$ to $T_\xi^{l,n}$. It is not hard to calculate that the cone of supporting normals to $T_\xi^{l,n}$ at any point $x^0 \in \partial T_\xi^{l,n}$ is at most two-dimensional. Assuming that the vectors $\lambda \in K_e$ are always normalized, $|\lambda| = 1$, form the quantity

$$\lambda^*(\xi) = \lim_{\eta \to \xi} \sup_{x \in \partial T_\xi^{l,n}} |\lambda^0(x)|.$$

If x is an arbitrary point of $T_\xi^{l,n}$, then

$$\operatorname{Re} \sum_{l+1}^{n} \overline{\lambda}_k (x_k^0 - x_k) \geq \left(\sum_{l+1}^{n} |\lambda_k|^2 \right)^{1/2} |x^0 - x| \min \left(\frac{1}{\sqrt{2}}, \frac{\beta n}{4\lambda^*(\xi)} |x^0 - x| \right) \quad (7)$$

(β is an absolute constant).

From this inequality follows the strict convexity of $T_\xi^{l,n}$, and it gives, so to speak, a quantitative characteristic of this strict convexity. We note that the fact of strict convexity also follows from Jenkins' theorem ($^{4-6}$) and the above-mentioned theorem of Schaeffer and Spencer stating that $S_x \cap S_n \neq \emptyset$ if $x \in \partial \mathcal{V}_n$. In turn, the theorem itself on quadratic functionals of the form $Q(w)(dw/w)^2$, generalizing Teichmüller' s theorem, can be obtained from the fact of strict convexity of the sections $T_\xi^{l,n}$, $l = [n/2]$. We note that in the work of Duren and Schiffer (^{7}), apparently, the possibility was first indicated of generalizing Teichmüller' s theorem in a formulation equivalent to the strict convexity of the sections under consideration.

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