ON QUASICONFORMAL HARMONIC MAPPINGS OF EUCLIDEAN SPACE

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Abstract

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ON QUASICONFORMAL HARMONIC MAPPINGS OF EUCLIDEAN SPACE

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Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a C^∞ mapping of n-dimensional Euclidean space $\mathbf{R}^n$ into itself, given by n coordinate functions $f_1(x), \dots, f_n(x)$, where $x = (x_1, \dots, x_n)$. The mapping f is called quasiconformal if the inequality

$$\left(\sum_{i,j=1}^{n} \left| \frac{\partial f_i}{\partial x_j} \right|^2 \right)^{1/2} \le C \, |J_f|^{1/n}, \tag{1}$$

holds everywhere, where C is some constant, and J_f is the Jacobian of the mapping f.

Combining two theorems of Liouville, one can prove the following assertion: any nonconstant quasiconformal mapping $f : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ which is at the same time holomorphic in the complex space $\mathbf{C}^n$, for $n > 1$, is a nondegenerate linear mapping.

In the present note we study a more general class of quasiconformal mappings $f : \mathbf{R}^n \to \mathbf{R}^n$ that are gradients of harmonic functions. Such mappings will be called harmonic.

Theorem. *A harmonic quasiconformal mapping $f : \mathbf{R}^3 \to \mathbf{R}^3$ that does not reduce to a constant is nondegenerate linear.*

Lemma 1. *If a quasiconformal mapping $f : \mathbf{R}^3 \to \mathbf{R}^3$ is given by analytic functions $f_i(x)$ $(i = 1, 2, 3)$ and is different from a constant, then it is a homeomorphism. The Jacobian J_f therefore cannot assume values of different signs.*

Let S be the set of zeros of J_f. As an analytic set, S is the union of a finite number of manifolds of class C^∞ (see ([1])). By virtue of inequality (1), the mapping f takes each connected component of S to a point. In the works ([2,3]) it is shown that the full preimage of a point under a quasiconformal mapping f, for any $\alpha > 0$, has zero α-dimensional Hausdorff measure. Consequently, each connected component of S consists of a single point, i.e. S is the union of a finite number of points. The set B_f of branch points of the mapping f belongs to S and, consequently, is also finite.

We show that $B_f = \emptyset$. If B_f is nonempty, surround any point $x \in B_f$ by such a neighborhood U that it contains no other points of B_f. It is known (see ([2],[3])) that $f(U)$ is a certain neighborhood of the point $f(x)$. On the one hand, the fundamental group $\pi_1(f(U) \setminus f(x))$ is trivial; on the other hand, in $f(U)$ there certainly are points different from $f(x)$ whose preimage consists of more than one point, which indicates the nontriviality of $\pi_1(f(U) \setminus f(x))$. This contradiction proves that $B_f = \emptyset$.

Thus, the mapping f is locally homeomorphic, and then, by the theorem of V. A. Zorich ([4]), it is homeomorphic globally as well. Finally, it is well known that under these conditions $f(\mathbf{R}^3) = \mathbf{R}^3$ (see ([5]) or ([6])). The lemma is proved.

It remains to show that all the functions $f_i(x)$ $(i = 1, 2, 3)$ in the formulation of the theorem are polynomials. Indeed, consider the mapping–

the mapping $h = \sigma f \sigma$, where $\sigma(x) = x|x|^{-2}$ is inversion. Since $f(\infty) = \infty$, we have $h(0) = 0$, and, by the multidimensional analogue of Mori' s theorem ([7]), there exist constants C_1 and C_2 such that, in some neighborhood of zero,

$$|h(x)| > C_1 |x|^{C_2}.$$

Returning to the mapping f, we obtain that for sufficiently large $|x|$

$$|f(x)| < \frac{1}{C_1} |x|^{C_2}.$$

This inequality is certainly also valid if in it $|f(x)|$ is replaced by $|f_i(x)|$ $(i = 1, 2, 3)$. Since all f_i are harmonic functions, it follows from the inequalities obtained that they are polynomials.

It also follows easily from inequality (1) that all degrees of the polynomials f_i are the same. We shall denote the common value of these degrees by m. We may write

$$f_i = p_i + q_i \qquad (i = 1, 2, 3),$$

where p_i are homogeneous polynomials of degree m, while the degrees of q_i are strictly less than m. It is obvious that the mapping $p : \mathbf{R}^3 \to \mathbf{R}^3$ defined by the coordinate functions p_i $(i = 1, 2, 3)$ is harmonic.

Lemma 2. Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be a quasiconformal homeomorphism, with $f(0) = 0$. Denote by $m(R)$ and $M(R)$, respectively, the minimum and maximum of $|f(x)|$ on the sphere $|x| = R$. Then there exists a constant C, depending only on f, such that

$$M(R) \leq Cm(R). \tag{2}$$

The proof follows easily from the estimate given by Gehring in ([8]) for the modulus of a ring.

Lemma 3. Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be a homeomorphism given by polynomials, and let $f(0) = 0$. Take some ray l issuing from the origin 0, and a point $x \in l$ such that $|x| = r$. If $s_l(r)$ is the length of the image of the segment $[0, x]$ under the mapping f, then there exists a constant $C(l)$, depending possibly on l, such that for every $r > 1$

$$s_l(r) \leq C(l)|f(x)|. \tag{3}$$

The proof is elementary.

Without loss of generality we may assume that for the original mapping f, $J_f \geq 0$ and $f(0) = 0$. Let $J_f|_l$ be the restriction of the Jacobian J_f to some ray l issuing from the origin. Then, in the notation of Lemma 3, $J_f|_l$ is a polynomial in r of degree k_l.

The number k_l does not depend on l. Indeed, for rays l_j $(j = 1, 2)$ such that $k_{l_1} < k_{l_2}$, take points x_j lying on l_j at distance $r > 1$ from the origin $(j = 1, 2)$. Using (2) and (3), we obtain

$$s_{l_1}(r) \geq m(r) \geq CM(r) \geq C|f(x_2)| \geq \frac{C}{C(l_2)}\, s_{l_2}(r).$$

But

$$s_{l_j}(r) = \int_0^r \mu_{l_j}(t)\, dt \qquad (j = 1, 2),$$

where $\mu_l(t)$ are the coefficients of local stretching in the corresponding directions. From the geometric definition of quasiconformality there follows the existence of constants C_1 and C_2 such that

$$C_1 \left(J_f|_{l_j}\right)^{1/3} \leq \mu_{l_j} \leq C_2 \left(J_f|_{l_j}\right)^{1/3} \qquad (j = 1, 2).$$

Hence $k_{l_1} \geq k_{l_2}$, which contradicts the inequality $k_{l_1} < k_{l_2}$.

Denote by k the common value of the k_l, and let us show that $k = 3(m - 1)$. On the sphere $|x| = 1$ there exists a point x such that $p(x) \neq 0$. It is easy to find constants $M_1(x)$ and $M_2(x)$ such that, for $r > 1$,

$$M_1(x) r^m \leq |f(rx)| \leq M_2(x) r^m. \tag{4}$$

If l is the ray determined by the vector x, then, using (3) and (4), we obtain

$$\int_0^r (J_f|_l)^{1/3}\,dt \geq \frac{1}{C_2}|f(rx)| \geq \frac{M_1(x)}{C_2}r^m = \frac{mM_1(x)}{C_2}\int_0^r [t^{3(m-1)}]^{1/3}\,dt.$$

It follows that $k \geq 3(m-1)$. Similarly one proves that $k \leq 3(m-1)$.

Let us show that everywhere on the unit sphere the Jacobian J_p of the mapping p, defined above by the homogeneous components of highest degree of the polynomials f_i $(i = 1, 2, 3)$, does not vanish. Indeed, if at some point x, $|x| = 1$, one had $J_p(x) = 0$, then

$$\lim_{r\to\infty} \frac{J_f(rx)}{r^{3(m-1)}} = \lim_{r\to\infty} \det\left(\frac{\partial p_i/\partial x_j|_{rx} + \partial q_i/\partial x_j|_{rx}}{r^{m-1}}\right) = \lim_{r\to\infty} \det\left(\left.\frac{\partial p_i}{\partial x_j}\right|_x + \frac{\partial q_i/\partial x_j|_{rx}}{r^{m-1}}\right) = 0,$$

and this contradicts the equality $k = 3(m-1)$.

In view of the compactness of the unit sphere, there exists a constant C such that, for $|x| = 1$,

$$\left(\sum_{i,j=1}^{3}\left|\frac{\partial p_i}{\partial x_j}\right|^2\right)^{1/2} \leq C|J_p|^{1/3}. \tag{5}$$

Since on the right and on the left in (5) there are homogeneous functions of degree $m-1$, this inequality is valid everywhere in $\mathbf{R}^3$, i.e., p is quasiconformal.

By Lemma 1 the mapping p is a homeomorphism, but, according to a result of Levy [9], the Jacobian of a harmonic homeomorphism in $\mathbf{R}^3$ does not vanish at 0. This is possible only when the degree of homogeneity of J_p is zero, i.e., $m = 1$. The theorem is proved.

Remark. All the results presented, except Levy' s theorem [9], are also valid for the n-dimensional $(n \geq 3)$ case. The question of the nonvanishing of the Jacobian of a harmonic homeomorphism in $\mathbf{R}^n$ $(n > 3)$ remains open.

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References

1. R. Thom, in: Collection, *Singularities of differentiable mappings*, 1968.

2. Yu. G. Reshetnyak, *Siberian Mathematical Journal*, 9, No. 2 (1968).

3. Yu. G. Reshetnyak, ibid., 8, No. 3 (1967).

4. V. A. Zorich, *Mathematical Collection*, 74 (116), No. 3 (1967).

5. B. V. Shabat, *Dokl. Akad. Nauk SSSR*, 132, No. 5 (1960).

6. Ch. Loewner, *J. Math. Mech.*, 8 (1959).

7. P. Caraman, *Homeomorfisme cvasiconforme n-dimensionale*, București, 1968.

8. F. W. Gehring, *Trans. Am. Math. Soc.*, 101, No. 3 (1961).

9. H. Levy, *Ann. Math.*, 88, No. 3 (1968).

Note: Figure translations are in progress. See original paper for figures.