QUASIELLIPTIC EQUATIONS IN AN INFINITE CYLINDER

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QUASIELLIPTIC EQUATIONS IN AN INFINITE CYLINDER

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1. Introduction. In the present article we study the solvability of a certain class of nonstationary differential equations in a cylinder infinite in time. Typical examples of such equations are the classical Laplace equation and the direct and inverse heat-conduction equations, and from this point of view the consideration of problems on an infinite time interval is natural.

Such problems (physicists call them problems without initial conditions) arise, for example, in the natural problem of the propagation of temperature in soil (see (^1)). In this case a unique solution is singled out by its belonging to a certain class that ensures a definite behavior for sufficiently large (in modulus) values of time (for example, boundedness of the solution as $t \to \pm\infty$).

We also proceed from this conception and prove unique solvability (for equations with coefficients independent of time) and normal solvability (in the general case) in S. L. Sobolev spaces with exponential behavior (of prescribed type) at infinity. Let us note that in the "Volterra" case (for example, for parabolic equations) one can prove unique solvability also for equations with coefficients depending on time; in this case, of course, as for normal solvability, it is necessary to impose certain requirements on the behavior of the coefficients as $t \to \pm\infty$.

In this article we study problems of S. L. Sobolev type (see (^2, ^3)). At the same time, the scheme of the proof of solvability is essentially of a "Banach" character. We note in this connection that related questions for the case of a cylinder with boundary were obtained in the works (^4, ^5).

2. Functional spaces. Let X be an n-dimensional smooth compact manifold without boundary and $\mathbf{R}^1$ the line. In the direct product—the cylinder $C = X \times \mathbf{R}^1$—introduce coordinates (x, t), $x \in X$, $t \in \mathbf{R}^1$. Then any function f on the (noncompact) manifold C is a function of the coordinates (x, t): $f = f(x, t)$. Now consider the complex plane $\mathbf{C}$. Choose on it the real line $\operatorname{Re} z = a$, parallel to the imaginary axis τ $(-\infty < \tau < +\infty)$, and fix it.

We now define, for a triple of real numbers (s, γ, α), the space $\hat{H}_{s,\gamma,\alpha}(C)$ as the space of distributions $f(x, t)$ on the cylinder $C = X \times \mathbf{R}^1$ with finite norm

$$\|f\|_{s,\gamma,\alpha} = \int_{\alpha-i\infty}^{\alpha+i\infty} \left\|(1+|z|^{2/\gamma}+\Delta)^{s/2}\hat{f}(z,x)\right\| \, |dz|.$$

Here Δ is a positive Laplace operator on the manifold X, constructed with the aid of some Riemannian metric, which from this moment is regarded as fixed, $\|\cdot\|$ is the L_2-norm on the manifold X, and

$$\hat{f}(z,x) = \int_{-\infty}^{+\infty} e^{zt} f(t)\, dt \qquad \operatorname{Re} z = \alpha.$$

We now introduce the space H_{s,γ,α_+} for a real quadruple of numbers $(s,\gamma,\alpha_+,\alpha_-)$. First let $k = s/\gamma$ be a natural number. Consider the piecewise constant function

$$\alpha(t) = \begin{cases} \alpha_+, & t > 0, \\ \alpha_-, & t < 0. \end{cases}$$

Then the space $H_{s,\gamma,\alpha_+,\alpha_-} = H_{s,\gamma,\alpha(t)}$ is the closure of smooth functions decreasing as $t \to \pm\infty$ with respect to the norm

$$\|f\|^2_{s,\gamma,\alpha_+,\alpha_-} = \int_{-\infty}^{+\infty} e^{2\alpha(t)t} \left(\|f\|_s^2 + \|f_t^{(k)}\|^2\right) dt.$$

In order to define the space $H_{s,\gamma,\alpha_+,\alpha_-}$ for an arbitrary quadruple $(s,\gamma,\alpha_+,\alpha_-)$, we consider a smooth partition of unity of the space $\mathbb{R}^1$

$$1 \equiv \sum_{i=1}^{2} \varphi_i(t),$$

where

$$\varphi_1(t) = \begin{cases} 1, & t > 0, \\ 0, & t < -1; \end{cases} \qquad \varphi_2(t) = \begin{cases} 0, & t > 1, \\ 1, & t < 0. \end{cases} \tag{1}$$

This partition of unity gives rise to a decomposition of an arbitrary function $f(x,t)$ into two summands

$$f(x,t) \equiv \varphi_1 f + \varphi_2 f = F_1(x,t) + F_2(x,t),$$

and, as is not difficult to see, if the function $f(x,t) \in H_{s,\gamma,\alpha_+,\alpha_-}$ (s/γ is an integer), then $F_1(x,t) \in H_{s,\gamma,\alpha_+}$, $F_2(x,t) \in H_{s,\gamma,\alpha_-}$. Therefore the expressions written below are finite, and we may define the space $H_{s,\gamma,\alpha_+,\alpha_-}$ as the closure of smooth functions decreasing as $t \to \pm\infty$ with respect to the norm

$$\|f\|_{s,\gamma,\alpha_+,\alpha_-} = \|F\|_{s,\gamma,\alpha_+} + \|F\|_{s,\gamma,\alpha_-}.$$

It can be shown that different partitions of unity on the cylinder C lead to spaces with equivalent norms.

Below we shall use the spaces $H_{s,\gamma,\alpha_+,\alpha_-}$ in the case when one of the numbers α_+, α_- tends to infinity.

Let, for example, α_+ be some fixed finite number; then the spaces $H_{s,\gamma,\alpha_+,\alpha_-}$ with variable α_- may be filtered in the following way:

$$\cdots \supset H_{s,\gamma,\alpha_+,\alpha'_-} \supset H_{s,\gamma,\alpha_+,\alpha''_-} \supset \cdots,$$

where

$$\cdots \geq \alpha'_- \geq \alpha''_- \geq \cdots.$$

The space $H_{s,\gamma,\alpha_+,-\infty}$ is defined as the intersection

$$H_{s,\gamma,\alpha_+,-\infty} = \bigcap_{\alpha_-} H_{s,\gamma,\alpha_+,\alpha_-}$$

of all the spaces $H_{s,\gamma,\alpha_+,\alpha_-}$ and is endowed with the topology of the projective limit, while the space $H_{s,\gamma,\alpha_+,+\infty}$ is the union

$$H_{s,\gamma,\alpha_+,+\infty} = \bigcup_{\alpha_-} H_{s,\gamma,\alpha_+,\alpha_-}$$

of the spaces $H_{s,\gamma,\alpha_+,\alpha_-}$ and is endowed with the topology of the inductive limit.

The spaces for $\alpha_+ = \pm\infty$ are defined analogously.

3. We shall consider quasi-elliptic differential expressions. Let D be a differential expression on the cylinder C. In each local coordinate system $(t, x^1, \dots, x^n)$, the expression D is a polynomial in the variables $(\partial/\partial t, \partial/\partial x^1, \dots, \partial/\partial x^n)$ with infinitely differentiable coefficients defined in the coordinate system $(t, x^1, \dots, x^n)$. Allowing a certain freedom, we shall often write the expression D in the form

$$D = D(x, t, \mathcal{D}_x, \partial/\partial t),$$

putting into this notation the exact meaning discussed above.

Let now γ be some positive number. Define the senior (principal) part $D_0 = D_0(x, t, \mathcal{D}_x, \partial/\partial t)$ of the expression as the γ-homogeneous part with respect to the variables $\partial/\partial t$ and $\mathcal{D}_x$ of the expression D. This means that

$$D_0(x, t, \lambda^\gamma \partial/\partial t, \lambda \mathcal{D}_x) = \lambda^m D_0(x, t, \partial/\partial t, \mathcal{D}_x).$$

We now associate with the differential expression $D(x, t, \partial/\partial t, \mathcal{D}_x)$ the characteristic polynomial $D_0(x, t, -z, -i\xi)$ by means of the formal substitution

$$\partial/\partial t \to -z, \qquad \mathcal{D}_x \to -i\xi$$

in its senior part D_0.

Definition. The differential expression $D(x, t, \partial/\partial t, \mathcal{D}_x)$ is called **quasielliptic** if, at every point (x, t), the equation

$$D_0(x, t, -z, -i\xi) = 0$$

has no purely imaginary roots $z = z(t, x, i\xi)$ for $\xi \neq 0$.

Remark. If in the expression $D(x, t, \mathcal{D}_x, \partial/\partial t)$ one makes the formal substitution $\partial/\partial t \to -q'$, then we arrive at a family of expressions

$$D_q = D(x, t, -q', \mathcal{D}_x),$$

and the condition introduced above is equivalent to quasiellipticity in the sense of Agranovich–Vishik ([6]) of the family D_q on the line $\operatorname{Re} z = 0$. It also coincides with the condition of weighted ellipticity in the sense of Agmon–Nirenberg ([4]).

4. The C. L. Sobolev problem for quasielliptic equations. Let X be a smooth compact manifold without boundary and let X_p, $p = 1, \dots, l$, be its submanifolds of codimensions $\nu_p \geq 1$. We shall consider the cylinder $C = X \times \mathbf{R}^1$ and, in it, the cylindrical submanifolds $C_p = X_p \times \mathbf{R}^1$. Consider the C. L. Sobolev problem

$$D(x, t, \mathcal{D}_x, \partial/\partial t) u(x, t) \equiv f(x, t) \quad (\text{mod} \bigcup_p C_p), \tag{2}$$

$$B_{pj}(x, t, \mathcal{D}_x, \partial/\partial t) u(x, t) = g_{pj}(x_p, t) \quad \text{on } C_p, \tag{3}$$

$$p = 1, \dots, l, \quad j = c_{j_1, \dots, j_{\nu_p}}, \quad |j| = j_1 + \dots + j_{\nu_p} \leq \kappa_p,$$

$$\kappa_p = \begin{cases} m - s - \nu_p/2, & \text{if } m - s - \nu_p/2 \text{ is not an integer,} \\ m - s - \nu_p/2 - 1, & \text{if } m - s - \nu_p/2 \text{ is an integer,} \end{cases}$$

where $\deg D = m$, $\deg B_{pj} = b_{pj}$.

The notion of quasiellipticity of the C. L. Sobolev problem is introduced in the natural way.

In the usual way we associate with the problem (2), (3) the operator

$$(D, B) : H_{s,\gamma,\alpha}(C) \to H_{s-m,\gamma,\alpha_+,\alpha_-}(C)/ \bmod R \oplus \bigoplus_{p,j} H_{s-b_{pj}-\nu_p/2,\gamma,\alpha_+,\alpha_-}(C_p). \tag{4}$$

5. Main theorems.

Theorem 1 (finiteness). Let the coefficients of the differential expressions in the quasielliptic operator (4) be independent of the variable t. Then for any numbers s and any finite numbers α_+ and α_-, except for a certain discrete set, the operator (4) is almost an isomorphism. Moreover,

I. If $\alpha_- < \alpha_+$, the operator (4) is monomorphic.

II. If $\alpha_- > \alpha_+$, the operator (4) is epimorphic.

III. If $\alpha_- = \alpha_+$, the operator (4) is an isomorphism.

Here we use the terminology of N. Bourbaki, according to which an almost isomorphism is a homomorphic, i.e. continuous and with closed range, mapping with finite-dimensional (over the field C) kernel and cokernel.

In the case when one of the numbers α_- or α_+ is infinite, the following is valid.

Theorem 2 (on homomorphism). Under the assumptions of Theorem 1, for any numbers s and any α_+ and α_-, except for a certain discrete set, ···

are homomorphisms. Moreover:

$$(D, B) : H_{s,\gamma,\alpha_+,-\infty}(C) \to H_{s-m,\gamma,\alpha_+,-\infty}(C)/ \bmod R \oplus H_{s-b_{pj}-\nu_p/2,\gamma,\alpha_+,-\infty}(C_p), \tag{5}$$

$$(D, B) : H_{s,\gamma,+\infty,\alpha_-}(C) \to H_{s-m,\gamma,+\infty,\alpha_-}(C)/ \bmod R \oplus H_{s-b_{pj}-\nu_p/2,\gamma,+\infty,\alpha_-}(C_p), \tag{6}$$

$$(D, B) : H_{s,\gamma,\alpha_+,+\infty}(C) \to H_{s-m,\gamma,\alpha_+,+\infty}(C)/ \bmod R \oplus H_{s-b_{pj}-\nu_p/2,\gamma,\alpha_+,+\infty}(C_p), \tag{7}$$

$$(D,B): H_{s,\gamma,-\infty,\alpha_-}(C) \to H_{s-m,\gamma,-\infty,\alpha_-}(C)/\operatorname{mod} R \oplus H_{s-b_{pj}-\nu_p/2,\gamma,-\infty,\alpha_-}(C_p) \quad (8)$$

are homomorphisms. Moreover:

I. The operators (5), (6) are monomorphisms with, generally speaking, infinite-dimensional kernel.

II. The operators (7), (8) are epimorphisms with, generally speaking, infinite-dimensional cokernel.

The following theorem gives an asymptotic representation, as $t \to \pm\infty$, of the solution of the S. L. Sobolev problem. For simplicity we formulate it in the case $\alpha_+ = \alpha_- = \alpha$.

Theorem 3. Let the function $u(x,t) \in H_{s,\gamma,\alpha}(C)$, and let the functions $f(x,t) \in H_{s'-m,\gamma,\alpha_1}$, $g_{pj} \in H_{s'-b_{pj}-\nu_p/2,\gamma,\alpha_1}$, where $\alpha < \alpha'$, $s < s'$. Then in some tubular neighborhood of the submanifold C_p the solution $u(x,t)$ of problem (2), (3) can be represented in the form

$$u(x,t) = \sum_k \sum_{i=0}^{r_k-1} t^i e^{-z_k t} \left\{ \sum_q a_{ikq}(\omega) \rho^{m-\nu_p-\chi_p+q} + \right.$$

$$\left. + \left(1 + (-1)^{m-\chi_p-\nu_p+q}\right) \rho^{m-\nu_p-\chi_p+q} \ln \rho \right\} + u_1(x,t).$$

Here $a_{ikq}(\omega)$ are smooth functions in a tubular neighborhood of the manifold C_p; ρ is the distance from the point (x,t) to the submanifold C_p.

The outer summation is carried out over all poles $z_1, \dots, z_k$ with multiplicities $r_1, \dots, r_k, \dots$, lying in the strip $\alpha < \operatorname{Re} z < \alpha_1$, and the remainder $u_1(x,t) \in H_{s',\gamma,\alpha_1}(C)$.

Finiteness theorems and theorems on asymptotic expansion have also been obtained for equations with variable coefficients.

6. I take this occasion to indicate bibliographical references not noted by me in note (7). Boundary operators (additional potentials) were used earlier by M. I. Vishik and G. I. Eskin in a series of their joint works (see, for example, (8)). The Mellin transform method for the study of boundary-value problems in a plane domain with angular points was used earlier by G. I. Eskin (9, 10). In the course of work on paper (7), G. I. Eskin gave me the opportunity to become acquainted with the manuscript of paper (10).

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Note: Figure translations are in progress. See original paper for figures.