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Abstract

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MATHEMATICS

M. I. KANOVICH

THE COMPLEXITY OF DECIDING AN ENUMERABLE SET AS A CRITERION OF ITS UNIVERSALITY

(Presented by Academician A. A. Dorodnitsyn on 13 March 1970)

1. We shall use the terminology and concepts introduced in ([1−3]). In particular, the length of the representation of a Φ-algorithm $\mathfrak{A}$ will be called its **complexity** and denoted by the symbol $\mathfrak{A}\mathfrak{S}$.

Let $\mathfrak{M}$ be an enumerable set, and n a natural number. A Φ-algorithm $\mathfrak{B}$ will be called $(\mathfrak{M}, n)$-**deciding** if the algorithm $\mathfrak{B}$ is applicable to all natural numbers not exceeding n, and annuls precisely those of them which belong to the set $\mathfrak{M}$. Let f be a general recursive function. An $(\mathfrak{M}, n)$-deciding algorithm $\mathfrak{B}$ will be called $(\mathfrak{M}, n, f)$-**deciding** if, on any number x not exceeding n, the algorithm $\mathfrak{B}$ finishes its work in no more than $f(x)$ steps.

A Boolean vector R is called a **segment** of the enumerable set $\mathfrak{M}$ if

$$\forall x(x < [B^\circ \supset (\sigma_{x+1}(R) \doteq | \equiv x \in \mathfrak{M})).$$

A nonempty segment R of the set $\mathfrak{M}$ will be called a $([R^\circ - 1])$-**segment** of the set $\mathfrak{M}$.

Let $\mathfrak{M}$ be an enumerable set, and n a natural number. A Φ-algorithm $\mathfrak{C}$ will be called an $(\mathfrak{M}, n)$-**characteristic** if the algorithm $\mathfrak{C}$ is applicable to the empty word Λ, and the word $\mathfrak{C}(\Lambda)$ is an n-segment of the set $\mathfrak{M}$ (cf. ([4])). Let t be a natural number. An $(\mathfrak{M}, n)$-characteristic $\mathfrak{C}$ will be called an $(\mathfrak{M}, n, t)$-**characteristic** if $\mathfrak{C}$ finishes its work on the empty word in no more than t steps.

2. An enumerable set $\mathfrak{M}$ is called **effectively nonrecursive** if there exists an unbounded general recursive function f such that, whatever the natural number n, every $(\mathfrak{M}, n)$-deciding algorithm has complexity not less than $f(n)$.

An enumerable set $\mathfrak{M}$ will be called **strictly nonrecursive** if, for every partial recursive function φ, one can indicate an unbounded general recursive function g such that, whatever the natural number n, if φ is a strictly increasing general

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recursive function, then every $(\mathfrak{M}, n, \varphi)$-deciding algorithm has complexity not less than $g(n)$.

An enumerable set $\mathfrak{M}$ will be called **effectively complex** if there exists an unbounded general recursive function f such that, whatever the number n, every $(\mathfrak{M}, n)$-characteristic has complexity not less than $f(n)$.

An enumerable set $\mathfrak{M}$ will be called **strictly complex** if, for every partial recursive function φ, one can indicate an unbounded general recursive function g such that, whatever the number n, if φ is a strictly increasing general recursive function, then the complexity of every $(\mathfrak{M}, n, \varphi(n))$-characteristic is not less than $g(n)$.

We shall say that an enumerable set $\mathfrak{N}$ is **reducible** to an enumerable set $\mathfrak{M}$ **by weak tables** if there exist a Φ-algorithm $\mathfrak{A}$ and a general recursive function f such that, whatever the natural number n, the algorithm $\mathfrak{A}$ is applicable to every $f(n)$-segment R of the set $\mathfrak{M}$, and $\mathfrak{A}(R) \doteq \Lambda \equiv n \in \mathfrak{N}$ (cf. ([5])).

We shall call an enumerable set $\mathfrak{M}$ **universal with respect to weak tabular reducibility** if every enumerable set is reducible to the set $\mathfrak{M}$ by weak tables.

Theorem 1. *Let $\mathfrak{M}$ be an enumerable set. Then the following five propositions are pairwise equivalent: 1) the set $\mathfrak{M}$ is universal with respect to weak tabular reducibility; 2) the set $\mathfrak{M}$ is effectively nonrecursive; 3) the set $\mathfrak{M}$ is strictly nonrecursive; 4) the set $\mathfrak{M}$ is effectively complex; 5) the set $\mathfrak{M}$ is strictly complex.*

3. Let us consider questions connected with estimates of the complexity of bounded decision for enumerable sets that are universal in the sense of Turing.

3.1. First we shall give a number of definitions. By natural numbers we mean words of the form $0P$, where P is a word in the alphabet $|$. By the symbol $\sigma_i(R)$, where i is a positive number not exceeding the length of R, we shall denote the i-th letter of the word R. For convenience we shall assume that $\sigma_0(R) \rightleftarrows 0$.

Let $\mathfrak{F}$ be a Φ-algorithm, R a Boolean vector. We shall call a natural number y an $(\mathfrak{F}, R)$-consequence of the natural number x, if there exists a sequence of natural numbers $n_0, n_1, \dots, n_k, n_{k+1}, y_0, y_1, \dots, y_k, y_{k+1}$ such that: 1) $n_0 = n_{k+1} = 0,\ 1 \leq n_i \leq |R|$ $(i = 1, 2, \dots, k)$; 2) $y_0 = x,\ y_{k+1} = y$; 3) $\forall j(0 \leq j \leq k \to y_{j+1} n_{j+1} = \mathfrak{F}(y_j n_j \sigma_{n_j}(R)))$.

Let $\mathfrak{M}$ be an enumerable set, $\mathfrak{F}$ a Φ-algorithm. We shall call a natural number y an $\mathfrak{M}$-value of the algorithm $\mathfrak{F}$ on the natural number x (notation $y = \mathfrak{M}\mathfrak{F}(x)$), if it is false that no such segment R of the set $\mathfrak{M}$ is possible for which y is an $(\mathfrak{F}, R)$-consequence of the number x (cf. ([6])).

Introduce the following notation:

$$y \geqslant_{\mathfrak{M}\mathfrak{F}} (x) \rightleftarrows \neg\neg\exists z(y \geq z \ \&\ z = \mathfrak{M}\mathfrak{F}(x)),$$

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$$y \leqslant_{\mathfrak{M}\mathfrak{F}} (x) \rightleftarrows \neg\neg\exists z(y \leq z \ \& \ z = \mathfrak{M}\mathfrak{F}(x)),$$

$$!_{\mathfrak{M}\mathfrak{F}}(x) \rightleftarrows \neg\neg\exists y(y = \mathfrak{M}\mathfrak{F}(x)).$$

We shall call a Φ-algorithm $\mathfrak{F}$ an **$\mathfrak{M}$-recursive function** if $\forall x !_{\mathfrak{M}\mathfrak{F}}(x)$ (cf. ([6])). An $\mathfrak{M}$-recursive function $\mathfrak{F}$ will be called **unbounded** if $\forall y \neg\neg\exists x(y \leqslant_{\mathfrak{M}\mathfrak{F}} (x))$. An $\mathfrak{M}$-recursive function $\mathfrak{F}$ will be called **nondecreasing** if

$$\forall x \neg\neg\exists y(y \geqslant_{\mathfrak{M}\mathfrak{F}} (x) \ \& \ y \leqslant_{\mathfrak{M}\mathfrak{F}} (x+1)).$$

3.2. An enumerable set $\mathfrak{M}$ is called **weakly effectively nonrecursive** if there exists an unbounded $\mathfrak{M}$-recursive function $\mathfrak{F}$ such that, whatever natural number n is taken, for every $(\mathfrak{M}, n)$-deciding algorithm $\mathfrak{B}$ the condition $\mathfrak{B} \supset\geqslant_{\mathfrak{M}\mathfrak{F}} (n)$ is satisfied.

We shall call an enumerable set $\mathfrak{M}$ **weakly effectively complex** if there exists an unbounded nondecreasing $\mathfrak{M}$-recursive function $\mathfrak{F}$ such that, whatever natural number n is taken, for every $(\mathfrak{M}, n)$-characteristic $\mathfrak{C}$ the condition $\mathfrak{C} \supset\geqslant_{\mathfrak{M}\mathfrak{F}} (n)$ is satisfied.

We assume that we have a numbering of the Φ-algorithms $\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_k, \dots$. The algorithm with number k will be denoted by $\langle k \rangle$.

We shall call an enumerable set $\mathfrak{M}$ **τ-nonrecursive** if there exists an $\mathfrak{M}$-recursive function $\mathfrak{F}$ such that for any natural m and k such that $k = \mathfrak{M}\mathfrak{F}(m)$, the following conditions are satisfied: 1) the algorithm $\langle k \rangle$ is an unbounded $\mathfrak{M}$-recursive function; 2) if m is a Gödel number of a strictly increasing general-recursive function f, then, whatever n may be, for every $(\mathfrak{M}, n, f)$-deciding algorithm $\mathfrak{B}$ the condition $\mathfrak{B} \supset\geqslant_{\mathfrak{M}\langle k \rangle} (n)$ is satisfied.

We shall call an enumerable set $\mathfrak{M}$ **τ-complex** if there exists an $\mathfrak{M}$-recursive function $\mathfrak{F}$ such that for any m and k such that $k = \mathfrak{M}\mathfrak{F}(m)$, the following conditions are satisfied: 1) the algorithm $\langle k \rangle$ is an unbounded nondecreasing $\mathfrak{M}$-recursive function; 2) if m is a Gödel number of a strictly increasing general-recursive function f, then, whatever···

whatever n may be, for every $(\mathfrak{M}, n, f(n))$-characteristic $\mathfrak{C}$ the condition $\mathfrak{C} \supset\geqslant_{\mathfrak{M}\langle k \rangle} (n)$ is satisfied.

An enumerable set $\mathfrak{M}$ is called **Turing universal** if, for every enumerable set $\mathfrak{N}$, one can specify an $\mathfrak{M}$-recursive function $\mathfrak{F}$ such that $\forall n(n \in \mathfrak{N} \equiv 0| = \mathfrak{M}\mathfrak{F}(n))$ (see (5)).

Theorem 2. *An enumerable set is Turing universal if and only if it is τ-nonrecursive.*

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Theorem 3. *If an enumerable set is τ-complex, then it is false that it is not Turing universal.*

Theorem 4. *An enumerable set is Turing universal if and only if it is weakly effectively nonrecursive.*

Corollary 1. *Every Turing-universal set is weakly effectively complex.*

4. By the symbol $\overline{\mathfrak{M}}$ we shall denote the complement of the set $\mathfrak{M}$. An enumerable set $\mathfrak{M}$ will be called **τ-creative** if there is a Φ-algorithm $\mathfrak{F}$ such that, for every n, if n is the Gödel number of an enumerable subset $\mathfrak{N}$ of the set $\overline{\mathfrak{M}}$, then $!\mathfrak{M}\mathfrak{F}(n)$, and for every number k such that $k = \mathfrak{M}\mathfrak{F}(n)$ it is false that the intersection of the set $\{0, 1, \dots, k\}$ and the set $\overline{\mathfrak{M}} \setminus \mathfrak{N}$ is empty.

Corollary 2. *An enumerable set is Turing universal if and only if it is τ-creative (cf. (7) and Theorem 1 from (8)).*

5. Let f and g be general recursive functions. By the symbol $f \lhd g$ we shall denote the following assertion: "the function f is nondecreasing and $\forall n(f(n) \leqslant g(n))$." By the symbol $\chi(f)$ we shall denote the formula $\exists m \forall n(f(n) = 0 \vee f(n) = m)$. By the symbol $\mathfrak{M}_g$ we shall denote the set defined by the condition $n \in \mathfrak{M}_g \Leftrightarrow \exists x(g(x) = n)$.

The function g is called a **re-enumeration** of the enumerable set $\mathfrak{M}$ if $\forall n(n \in \mathfrak{M} \equiv n \in \mathfrak{M}_g)$. Let g be a general recursive function, and let $\mathfrak{M}$ be an enumerable set. We shall call the function g **$\mathfrak{M}$-regular** if there exists an $\mathfrak{M}$-recursive function $\mathfrak{F}$ such that $\forall nm(g(n) = g(m) \supset m \leqslant \mathfrak{M}\mathfrak{F}(n))$.

By the symbol $\eta(g)$ we shall denote the assertion: "there is a Φ-algorithm $\mathfrak{C}$ such that, for every number m which is the Gödel number of a general recursive function f such that $f \lhd g$ and $\chi(f)$, the condition $!_{\mathfrak{M}_g}\mathfrak{C}(m)$ & $\forall n(f(n) \leqslant \mathfrak{M}_g\mathfrak{C}(m))$ is satisfied."

Theorem 5. *For every re-enumeration g of a Turing-universal set $\mathfrak{M}$ one can specify an $\mathfrak{M}$-recursive function $\mathfrak{F}$ such that, for every number m which is the Gödel number of a general recursive function f such that $f \lhd g$, the condition $\forall n(f(n) \leqslant \mathfrak{M}\mathfrak{F}(m))$ is satisfied.*

Theorem 6. *If a general recursive function g is a re-enumeration of a τ-complex set $\mathfrak{M}$, then $\eta(g)$ holds.*

Theorem 7. *Let g be a re-enumeration of an enumerable set $\mathfrak{M}$. Let g be $\mathfrak{M}$-regular and let $\eta(g)$ hold. Then the set $\mathfrak{M}$ is weakly effectively complex.*

6. An enumerable set $\mathfrak{M}$ will be called **strongly nonrecursive** if, for every partial recursive function φ, one can specify such a nondecreasing general recursive function g that, if φ is a general recursive function, then: 1) the function g is unbounded; 2) whatever the natural number n may be, the complexity of every $(\mathfrak{M}, n, \varphi)$-deciding algorithm is not less than $g(n)$.

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Let f be a general recursive function. We shall say that the enumerable set $\mathfrak{M}$ **f-reduces** to the enumerable set $\mathfrak{N}$ if there exists a Φ-algorithm $\mathfrak{A}$ such that, whatever the natural number m may be, the following conditions are satisfied: 1) if $f(m) \in \mathfrak{N}$, then $!\mathfrak{A}(m0|)$ and $\mathfrak{A}(m0|) \doteq \Lambda \equiv m \in \mathfrak{M}$; 2) if $f(m) \notin \mathfrak{N}$, then $!\mathfrak{A}(m0|)$ and $\mathfrak{A}(m0|) \doteq \Lambda \equiv m \in \mathfrak{M}$.

By the letter $\mathfrak{L}$ denote the general recursive function defined by the equality $\mathfrak{L}(n) = [\log_2(n+1)]$.

Theorem 8. *For every enumerable set $\mathfrak{N}$ one can specify an enumerable set $\mathfrak{M}$ such that: 1) the set $\mathfrak{M}$ is $\mathfrak{L}$-reducible to the set $\mathfrak{N}$; 2) if the set $\mathfrak{N}$ is nonrecursive, then the set $\mathfrak{M}$ is strongly nonrecursive.*

Theorem 8 points to the impossibility of any substantial strengthening of Theorem 1. Theorem 8 also makes it possible to detect a discrepancy between table reducibility (see the definition in [9]) and weak table reducibility, using a method different from that of the paper [10].

Theorem 9. *Let a strongly nonrecursive set $\mathfrak{M}$ be table reducible to a set $\mathfrak{N}$. Then the set $\mathfrak{N}$ is strongly nonrecursive.*

By the letter E denote the general recursive function defined by the equality $E(n) = n$.

Theorem 10. *Whatever the hyperimmune set $\mathfrak{M}$, the unbounded general recursive function f, and the natural number m, one can specify a natural number n, exceeding m, such that it is not true that an $(\mathfrak{M}, n, E)$-deciding algorithm of complexity not exceeding $f(n)$ is impossible.*

Theorems 8–10 give the following corollary (cf. § 1.4 of the paper [10]):

Corollary 3. *For every hyperimmune set $\mathfrak{N}$ one can specify an enumerable set $\mathfrak{M}$ such that: 1) the set $\mathfrak{M}$ is $\mathfrak{L}$-reducible to the set $\mathfrak{N}$; 2) it is not true that $\mathfrak{M}$ is table reducible to the set $\mathfrak{N}$.*

7. In order to show the impossibility of any substantial strengthening of Theorem 2, we note that an enumerable set $\mathfrak{M}$ is nonrecursive if and only if for any partial recursive function φ one can specify a nondecreasing $\mathfrak{M}$-recursive function $\mathfrak{F}$ such that, if φ is a general recursive function, then: 1) $\mathfrak{F}$ is an unbounded $\mathfrak{M}$-recursive function; 2) for every natural number n and every $(\mathfrak{M}, n, \varphi)$-deciding algorithm $\mathfrak{B}$, the condition $\mathfrak{B} \supseteq_{\mathfrak{M}} \mathfrak{F}(n)$ holds.

In connection with Theorems 1 and 3, let us also note that, for any enumerable set $\mathfrak{M}$ and any general recursive function φ, one can specify an unbounded general recursive function g such that, whatever n may be, every $(\mathfrak{M}, n, \varphi(n))$-characteristic has complexity not less than $g(n)$.

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Moscow State University
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Note: Figure translations are in progress. See original paper for figures.

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