ON THE TRAJECTORIES OF SYSTEMS WITH A RIGHT-HAND SIDE OF EXTREMAL FORM

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Abstract

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Systems of differential equations of the form

$$\dot{x}_i = \sum_{j=1}^{m} a_{ij} \min_k b_{jk} x_k, \qquad i, k = 1, 2, \dots, n \tag{1}$$

are considered.

Systems of this kind may be arrived at, at least, from two standpoints. First of all, such systems describe processes for which the presence of a so-called “bottleneck” is essential. This question is discussed in detail in (^{1}). In addition, similar systems (as systems for dual variables) arise in some problems of optimal control. For the special case of system (1), when a_{ij} is the Kronecker symbol, this question is considered in (^{2}).

In the present communication we shall consider some properties of system (1), restricting ourselves to the case of two equations.

For $n = 2$, system (1) can be reduced to the form

$$\dot{x}_i = \sum_{j=1}^{m} \left[a_{ij}^{(1)} \min_k b_{jk} x_k + a_{ij}^{(2)} \max_k b_{jk} x_k \right], \qquad i, k = 1, 2, \tag{2}$$

where $b_{j2} \geq 0$, while b_{j1} and $a_{ij}^{(k)}$ are arbitrary real numbers. We shall assume that b_{j1} and b_{j2} do not vanish simultaneously for any j.

If, in particular, $a_{ij}^{(1)} = a_{ij}^{(2)}$, then system (2) becomes a linear homogeneous system with constant coefficients. In the general case the following holds.

Theorem 1. The pencil of straight lines $b_{j2} x_2 = b_{j1} x_1$, $j = 1, 2, \dots, m$, divides the plane $x_1 O x_2$ into $2m$ angles, in each of which system (2) becomes a linear homogeneous system with constant coefficients.

Corollary 1. The rays issuing from the origin are isoclines for the trajectories of system (2).

Corollary 2. For arbitrary matrices $\|a_{ij}^{(1)}\|$, $\|a_{ij}^{(2)}\|$, $\|b_{ij}\|$, the origin is a stationary solution of system (2). Other stationary solutions, if they exist, fill entirely either some of the $2m$ angles, or rays issuing from the origin.

Corollary 3. If the point (x_1^0, x_2^0) is situated outside the set of singular points indicated in Corollary 2, then the solution of the Cauchy problem for system (2) with initial data x_1^0, x_2^0 can be obtained in explicit form without quadratures.

Let us note that the existence and uniqueness of such a solution follows from the fact that the right-hand sides of system (2) satisfy the Lipschitz condition in both variables.

Suppose now that, apart from the origin, system (2) has no stationary solutions. For this it is necessary and sufficient that for every unit vector h_1, h_2 there be found at least one value $i = 1, 2$ such that

$$\sum_{j=1}^{m} \left(a_{ij}^{(1)} \min_k b_{jk} h_k + a_{ij}^{(2)} \max_k b_{jk} h_k \right) \neq 0.$$

Consider the question of the behavior of trajectories in a neighborhood of the origin in this case.

Proposition 1. Let A_k and A_{k+1} be the matrices of system (2) in the k-th and $(k+1)$-st angles, respectively. Then the equality

$$A_{k+1} = A_k + \widetilde{A},$$

holds, where $\widetilde{A}$ is a matrix of the form

$$\| -\alpha_i \quad \beta\alpha_i \|.$$

Proposition 2. The matrix of system (2) in the k-th angle is uniquely determined if the matrices in the $(k-1)$-st and $(k+1)$-st angles are given.

Suppose now that there are two systems

$$\dot{X} = AX, \qquad A = \|a_{ij}\|, \qquad i, j = 1, 2; \tag{3}$$

$$\dot{X} = (A + \widetilde{A})X, \qquad \widetilde{A} = \| -\alpha_i \quad \beta\alpha_i \|. \tag{4}$$

We shall call the matrix A (or $A + \widetilde{A}$) **specially degenerate** if it has zero rank, or if the ratio of the elements of the second column to the elements of the first column is equal to $-\beta$.

Proposition 3. If one of the matrices A or $A+\widetilde{A}$ is specially degenerate, then the other of these matrices is also specially degenerate.

Proposition 4. If A is specially degenerate, then, in addition to the origin, system (3) has infinitely many stationary solutions.

Proposition 5. If $\gamma = a_{21}\beta^2 - \beta(a_{11} - a_{22}) - a_{12} \neq 0$, then the matrix A is not specially degenerate.

Lemma 1. If $\gamma \neq 0$, then, whatever the matrix A may be, the point (α_1, α_2) in the plane $\alpha_1 0 \alpha_2$ can be chosen so that the origin for system (4) will be a singular point of any prescribed type. To the exceptional directions (in the case of a node and a saddle), by the choice of (α_1, α_2), one can assign any prescribed angular coefficient except $1/\beta$. Finally, (α_1, α_2) can be chosen so that the trajectories of system (4) will be parallel straight lines.

Let us also note that a saddle and a node can be obtained by choosing α_1, α_2 of the desired signs. This is important in describing irreversible processes.

Lemma 2. If $\gamma = 0$ and the matrix A is not specially degenerate, then the matrix $A + \widetilde{A}$, just as the matrix A, has only real eigenvalues. In this case one of the exceptional directions for both systems (3) and (4) has angular coefficient equal to $1/\beta$.

A consequence of the stated propositions and lemmas is:

Theorem 2. Let the plane $x_1 0 x_2$ be divided by a pencil of straight lines passing through the origin into $2m$ angles. Suppose that in each of the $2m-1$ angles the type of the singular point $(0,0)$ is prescribed in such a way that the boundary ray for the trajectories of the adjacent angles separated by it is, or is not, simultaneously an exceptional direction. Then there exists a system (2) whose trajectories will behave in accordance with the prescribed combination of types.

Theorem 2 makes it possible to construct systems (2) with singular points of various types, such as, for example, a multiple saddle, a multiple node, a multiple saddle-node, and so on. The indices of these multiple points are greater than one in absolute value.

The statement converse to Theorem 2 is true: the combinations of trajectories of linear systems indicated in its hypotheses exhaust all possibilities for the arrangement of trajectories of systems (2) in a neighborhood of the origin (under the condition that $(0,0)$ is an isolated singular point).

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Note: Figure translations are in progress. See original paper for figures.

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