NORMALIZATION OF THE WIENER–HOPF OPERATOR

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 519.55

MATHEMATICS

V. B. DYBIN

NORMALIZATION OF THE WIENER–HOPF OPERATOR

(Presented by Academician N. I. Muskhelishvili, September 1, 1969)

We consider the operators

$$A\varphi \equiv \left\{ \sum_{k=0}^{\infty} a_{n-k}\varphi_k \right\}_{n=0}^{\infty}, \tag{1}$$

$$(\lambda I - K)\varphi \equiv \lambda\varphi(t) - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} k(t-s)\varphi(s)\, ds, \qquad t > 0, \tag{2}$$

in the so-called exceptional case, when their symbols

$$a(t) = \sum_{n=-\infty}^{\infty} a_n t^n, \qquad \lambda - \mathcal{K}(x) = \lambda - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(t)e^{ixt}\, dt$$

vanish on the unit circle or, respectively, on the real axis. In this connection we note a series of papers ($^{1-10}$), in which a number of cases of the indicated operators are investigated, and above all the case of zeros of power type.

Below the operators (1) and (2), admitting factorization, are studied under the assumption that their symbols have on the contour a finite number of arbitrary zeros of finite order. For them a theory of normal solvability is constructed, the index is computed, and new exceptional cases are presented.

1. Normalization of linear operators with nonclosed range. A linear (bounded) operator A in a Banach space X is called a Φ-operator*, if its range $\operatorname{im} A$ is closed in X, and the d-characteristic $(\alpha_A, \beta_A) = (\dim\ker A, \dim\ker A^*)$ is finite. The index of the operator A is the number $\varkappa_A = \alpha_A - \beta_A$.

If the operator A is not a Φ-operator, then by its normalization we shall mean the construction of a Φ-operator $\widetilde{A}$, differing from A by the assignment of a new space of ranges**.

Definition. We shall say that a linear operator $A(X \to X)$ admits a **normalization** if there exists a Banach space DX, $\operatorname{im} A \subseteq DX \subseteq X$, such that the operator $\widetilde{A}(X \to DX)$, $\widetilde{A}\varphi = A\varphi$ for all $\varphi \in X$, is a Φ-operator.

If $\alpha_A < \infty$, then the operator A always admits a normalization. In this case the choice of the space DX can be made, generally speaking, in a nonunique way. As DX one may take the space $\operatorname{im} A \dot{+} L$, $L \subset X \setminus \operatorname{im} A$, with the corresponding renorming.

Definition. By the **order** of a normalization of the operator A we shall mean the number $\dim \ker \widetilde{A}^*$. A normalization of the operator A will be called **proper** if $\ker \widetilde{A}^* \subseteq \ker A^*$. Among all proper normalizations possible for the operator A, the **maximal** one will be that which has the highest order.

* Another common name is a Noether operator, i.e. one satisfying the theorem of F. Noether (see, for example, ([11])).

** In this connection see also ([7,8]).

In order that the operator A admit a maximal normalization, it is necessary and sufficient that its d-characteristic be finite. If $\beta_A = 0$ $(\alpha_A < \infty)$, then such a normalization is unique $(DX = \operatorname{im} A)$. If $\beta_A \neq 0$, then the maximal normalization is carried out in a nonunique way, but always $\ker \tilde{A} = \ker A$, $\ker A^* = \ker A^*$.

2. Discrete Wiener-Hopf operator. Consider the Banach algebra W of functions $a(t)$, continuous on the unit circle, that expand into an absolutely convergent Fourier series,

$$a(t) = \sum_{k=-\infty}^{\infty} a_k t^k.$$

In W we shall distinguish the subalgebras $W^{\pm}$ of functions, respectively, of the form

$$a^{+}(t) = \sum_{k=0}^{\infty} a_k t^k \quad \text{and} \quad a^{-}(t) = \sum_{k=-\infty}^{0} a_k t^k.$$

The algebra W generates the linear manifold $\mathfrak{W}$ of Wiener-Hopf operators of the form (1). If

$$\sum_{k=-\infty}^{\infty} |a_k| < \infty,$$

then the operator A is defined and bounded in the spaces l_{p+} $(1 \leq p \leq \infty)$, c_{+}^{0} *, and its symbol belongs to the algebra W. The algebras $W^{\pm}$ generate commutative algebras $\mathfrak{W}^{\pm} \subset \mathfrak{W}$ of operators $A_{\pm}$ of the form (1), whose symbols $a^{\pm}(t) \in W^{\pm}$. The spaces of maximal ideals of the algebras $W, W^{\pm}$ are described as follows (([12]), pp. 31-32): a) the set of all functions $a(t) \in W$ satisfying the condition $a(t_0) = 0$, $|t_0| = 1$, forms a maximal ideal J_{t_0} of the algebra W; b) the set of all functions $a^{\pm}(t) \in W^{\pm}$ satisfying the condition $a^{\pm}(z_0) = 0$, $|z_0| \leq 1$ $(|z_0| \geq 1)$, forms a maximal ideal $J_{z_0}^{\pm}$ of the algebra $W^{\pm}$; the indicated ideals exhaust all maximal ideals of the algebras $W, W^{\pm}$. Those ideals $J_{z_0}^{\pm}$ of the algebras $W^{\pm}$ that correspond to points $|z_0| < 1$ ($|z_0| > 1$) will henceforth be called **essential**, and the remaining maximal ideals of these algebras—**inessential**. The essential ideals $J^{\pm}_{z_0}$ admit a simple description in terms of the volume of the functions contained in them: if $a^{\pm}(t) \in J^{\pm}_{z_0}$, $|z_0| < 1$ ($|z_0| > 1$), then there exists a natural number $k_{\pm}$ such that

$$a^{\pm}(t) = (t^{\pm 1} - z_0)^{k_{\pm}} a_1^{\pm}(t),$$

where $a_1^{\pm}(z_0) \neq 0$. If $A_{\pm} \in \mathfrak{W}^{\pm}$ and its symbol $a^{\pm}(t) \in J^{\pm}_{z_0}$, where $|z_0| < 1$ ($|z_0| > 1$), then the above number $k_{\pm}$ will be called the **characteristic** of the operator $A_{\pm}$ (or of its symbol $a^{\pm}(t)$) at the point z_0.

Denote by $\psi(z_0) = \{\psi_k\}_{k=0}^{\infty}$ the following vector from l_+: $\psi_0 = -z_0$, $\psi_1 = 1$, $\psi_k = 0,\ k > 1$, generated by the function $t - z_0 \in W^+$. By $[\psi(z_0)]^m$ denote the m-fold convolution

$$\psi(z_0) * \psi(z_0) * \cdots * \psi(z_0),$$

i.e. the vector of Fourier coefficients with nonnegative indices of the function $(t - z_0)^m \in W^+$. We give the definition of the characteristic of the operator $A_{\pm}$ at the point t_0, $|t_0| = 1$, corresponding to an inessential maximal ideal $J^{\pm}_{t_0}$. This characteristic depends, generally speaking, on the space in which the operator $A_{\pm}$ is considered.

Definition. Let $A_+ \in \mathfrak{W}^+$, and let its symbol $a^+(t) \in J^{\pm}_{t_0}$, $|t_0| = 1$, and $a^+(t) \in J^+_z$, $|z| \leq 1$, $z \neq t_0$. If

$$[\psi(t_0)]^m \in \operatorname{im} A_+ = A_+(l_{p+}),$$

but

$$[\psi(t_0)]^{m-1} \notin \operatorname{im} A_+,$$

then we shall say that the operator A_+ (or its symbol $a^+(t)$) has, at the point t_0, an l_p-characteristic equal to m. Let $A_- \in \mathfrak{W}^-$, and let its symbol belong to only one maximal ideal $J^-_{t_0}$, $|t_0| = 1$. We shall say that the operator A_- (or its symbol $a^-(t)$) has at the point t_0 an l_p-characteristic equal to m, if its transposed operator

$$A_-^{\tau} = A_+ \in \mathfrak{W}^+$$

has at the point

$$\bar{t}_0 = t_0^{-1}$$

an l_p-characteristic equal to m. The definition of the l_p-characteristic extends in a natural way to the case of operators $A_{\pm}$ whose symbols belong to only a finite number of maximal ideals of the algebras $W_{\pm}$.

* The elements of the indicated spaces are vectors $\varphi = \{\varphi_k\}_{k=0}^{\infty}$. For a detailed definition see (13).

An arbitrary operator $A \in \mathfrak{W}$ is a Φ-operator if and only if its symbol $a(t)$ belongs to none of the maximal ideals of the algebra W (^{13}). Moreover, when the indicated condition is fulfilled, the operator A admits a factorization, i.e., a representation in the form $A = A_- A_+$, where $A_\pm \in \mathfrak{W}^\pm$. If the symbol $a(t)$ of the operator A has the form

$$\prod_{k=1}^{n} (t - t_k)^{\alpha_k} \tilde{a}(t),$$

where $|t_k| = 1$, $\operatorname{Re} \alpha_k > 0$, $\tilde{a}(t) \neq 0$ on the unit circle, then the operator A admits a maximal normalization of finite order (^{6,9,10}).

By F we denote the class of operators $A \in \mathfrak{W}$ admitting a factorization $A = A_- A_+$. It is assumed here that the symbols $a^\pm(t)$ of the operators $A_\pm$ belong to no more than a finite number of maximal ideals of the algebras $W^\pm$ and, at all points corresponding to inessential maximal ideals, have finite l_p-characteristics.

Theorem 1. Let $A_- \in \mathfrak{W}^-$. In order that in the space $m_+ = l_\infty + \alpha_{A_-} = \dim \ker A_- < \infty$, it is necessary that the symbol $a^-(t)$ of the operator A_- belong only to a finite number of maximal ideals of the algebra W^-. If $A_- \in F$, then $\alpha_A < \infty$.

The validity of the theorem follows from the fact that if $a^-(t)$ belongs to inessential ideals $J^-_{t_k}$, $|t_k| = 1$, $k = 1, 2, \dots, n$, and to essential ideals $J^-_{z_l}$, $|z_l| > 1$, $l = 1, 2, \dots, m$, has finite l_p-characteristics at the points t_k, and at the points z_l characteristics equal to s_l, then the linear spans generated by the following systems of vectors:

$$\varepsilon_k = \{t_k^r\}_{r=0}^{\infty}, \quad 1 \leq k \leq n; \qquad \psi_{lj} = \{r(r{-}1) \dots (r{-}(r{-}j))\bar{z}_l^r\}_{r=0}^{\infty}, \quad 0 \leq j \leq s_l{-}1,\ 1 \leq l \leq m,$$

belong to $\ker A_-$. If, moreover, $a^-(t)$ belongs to no other maximal ideals of the algebra W^-, then the direct sum of the indicated spans coincides with the subspace $\ker A_-$.

Corollary 1. In order that $A_\pm \in \mathfrak{W}^\pm$ admit in the spaces l_{p+} $(1 \leq p < \infty)$, c_+^0 a maximal normalization of finite order, it is necessary, and if $A_\pm \in F$, also sufficient, that its symbol $a^\pm(t)$ belong only to a finite number of essential maximal ideals of the algebra $W^\pm$.

Corollary 2. Let $A_\pm \in F$ and let $\widetilde{A}_\pm$ be the Φ-operator generated by the maximal normalization of the operator $A_\pm$. Then

$$\chi_{\widetilde{A}_+} = -\left(n + \sum_{l=1}^{m} s_l\right) \text{ in } l_+ \quad \text{and} \quad \chi_{\widetilde{A}_+} = -\sum_{l=1}^{m} s_l \text{ in } l_{p+}\ (p \geq 1),\ c_+^0;$$

$$\chi_{\widetilde{A}_-} = -\chi_{\widetilde{A}_\tau^-} \text{ in all spaces } l_{p+}\ (p \geq 1),\ c_+^0.$$

Consider in the spaces l_{p+} $(1 \le p \le \infty)$, c_+^0 the operator $A = A\_A_+ \in F$. Let $m^\pm$ be the number of essential maximal ideals $J_{z_k^\pm}^\pm$ to which the symbols $a^\pm(t)$ of the operators $A_\pm$ belong; let $s_k^\pm$ be their characteristics at the points $z_k^\pm$; let $n^\pm$ be the number of inessential maximal ideals $J_{t_j^\pm}^\pm$ to which the functions $a^\pm(t)$ belong; and let $\alpha_j^\pm(p)$ be their l_p-characteristics at the points $t_j^\pm$. Introduce the following notation:

$$s^\pm = \sum_{k=1}^{m^\pm} s_k^\pm, \qquad \alpha^\pm(p) = \sum_{j=1}^{n^\pm} \alpha_j^\pm(p) \quad (p \ge 1 \text{ in the case of the space } l_{p+},\ p = 0$$

in the case of the space c_+^0; $p = \infty$ in the case of the space m_+). Under the indicated assumptions the following result holds:

Theorem 2. In the spaces l_{p+} $(1 \le p < \infty)$, c_+^0

$$\alpha_A = \max(s^- - s^+ - \alpha^+(p),\, 0).$$

In the spaces l_{p+} $(1 < p < \infty)$, c_+^0

$$\beta_A = \max(s^+ - s^- - \alpha^-(p'),\, 0), \qquad p' = p/(p-1);$$

in the space l_+

$$\beta_A = \max(s^+ + \tilde{n} - a^-(\infty),\, 0),$$

where $\tilde{n}$ $(0 \le \tilde{n} \le n^+)$ is the number of maximal ideals $J_{t_j}^+$ that differ from each of the maximal ideals $J_{t_j}^-$, $1 \le j \le n^-$.

It follows from Theorem 2 that the operator $A \in F$ always admits a maximal normalization of finite order. In those cases where the construction of the operator A^{-1} is known, it is possible to describe effectively the subspace $\operatorname{im} A$, and with it the space DX of images of Φ-operators $\tilde{A}$.

3. The Wiener-Hopf integral operator. If $k(t) \in L(-\infty, \infty)$, then the operator (2) is defined and bounded in the spaces $L_p(0, \infty)$ $(1 \le p \le \infty)$, $C^0(0, \infty)$. Its symbol $\lambda - \mathcal{K}(x)$ belongs to the Banach algebra R of functions continuous on the real axis and representable by an absolutely convergent Fourier integral, enlarged by adjoining constants by one dimension. $R^\pm$ are the subalgebras of R consisting of functions analytically continuable into the upper (lower) half-plane. By a factorization of the operator $\lambda I - K$ one understands its representation in the form $\lambda I - K = (\lambda\_I - K\_)(\lambda_+ I - K_+)$, where the operators $\lambda_\pm I - K_\pm$ have symbols $\lambda_\pm - \mathcal{K}^\pm(x) \in R^\pm$.

The real axis, completed by one infinitely remote point, and the upper (lower) closed half-plane are spaces of maximal ideals of the algebras R, R^+ (R^-), respectively. Analogously to the case of the discrete operator, essential and inessential maximal ideals of the algebras $R^\pm$, and the L_p-characteristics of the operators $\lambda_\pm I - K_\pm$, are defined. In this case the role of the vector $\psi(z_0)$ is played by the function

$$\psi_{z_0}(t) = \begin{cases} -i\sqrt{2\pi}\exp(-t)[1 + (z_0 i - 1)t], & \text{if } z_0 \neq \infty, \\ -i\sqrt{2\pi}\exp(-t), & \text{if } z_0 = \infty. \end{cases}$$

Its Fourier transform has the form

$$\Psi_{z_0}(x) = \begin{cases} (x - z_0)(x + i)^{-2}, & \text{if } z_0 \neq \infty, \\ (x + i)^{-1}, & \text{if } z_0 = \infty. \end{cases}$$

For the operator (2), admitting such a factorization that the symbols of the factor multipliers belong to only a finite number of maximal ideals of the algebras $R^\pm$ and have finite L_p-characteristics at points of the real axis, results hold analogous to Theorems 1, 2 and the corollaries following from them. The general scheme of the proof of these theorems is preserved, with the small change that in the case under consideration the space $L_p(0, \infty)$ is not a part of $M(0, \infty)$, and therefore the analogue of Theorem 1 must be proved separately for each space under consideration.

Rostov State University

Received
14 VII 1969

REFERENCES

1. F. D. Gakhov, V. I. Smagina, *Izv. AN SSSR, ser. matem.*, **26**, No. 3 (1962).
2. B. V. Kvedelidze, Tr. Tbilissk. matem. inst. AN GruzSSR, **23** (1956).
3. Yu. I. Cherskii, Doctoral dissertation, Tbilisi, 1964.
4. V. B. Dybin, *Izv. AN BSSR, ser. fiz.-matem. nauk*, No. 3 (1966).
5. V. B. Dybin, *Izv. AN ArmSSR, Matem.*, **2**, No. 4 (1967).
6. V. B. Dybin, N. K. Karapetyants, *Izv. vyssh. uchebn. zaved., Matem.*, No. 10 (65) (1967).

7. M. I. Khaikin, ibid., No. 3 (1967).

8. G. N. Chebotarev, ibid., No. 3 (1968).

9. G. N. Chebotarev, ibid., No. 10 (1967).

10. N. K. Karapetyants, Candidate dissertation, Rostov State University, 1969.

11. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.

12. I. M. Gelfand, D. A. Raikov, G. E. Shilov, *Commutative Normed Rings*, Moscow, 1960.

13. M. G. Krein, *UMN*, **13**, issue 5 (83) (1958).

Note: Figure translations are in progress. See original paper for figures.