ON THE INTERSECTION OF TOPOLOGIES

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 513.831

MATHEMATICS

V. P. ZOLOTAREV

ON THE INTERSECTION OF TOPOLOGIES

(Presented by Academician P. S. Aleksandrov on 15 IV 1970)

It is well known that under refinements the dimension of a space may both increase and decrease. Therefore it is of interest to know how dimension behaves under refinements of a special kind: let $\dim(X, \tau_1) = 0$ and $\dim(X, \tau_2) = 0$, where τ_1, τ_2 are different topologies on the set X, and $\tau = \tau_1 \cap \tau_2$ (a set U is open in the topology τ if and only if U is open both in the topology τ_1 and in the topology τ_2). What, then, can be said about the dimension of the space (X, τ)? It is clear, for example, that the topology of an interval can be represented as the intersection of two 0-dimensional topologies. For this it is enough to take, as a base of the topology τ_1, the family $\{[r, a)\}$, and as a base of the topology τ_2, the family $\{(b, r]\}$, where a, b, r are arbitrary rational numbers, $[r, a) = \{x \mid r \le x < a\}$ and $(b, r] = \{x \mid b < x \le r\}$.

In A. V. Arkhangel' skii' s seminar of P. S. Aleksandrov in Moscow the following hypothesis was formulated: $\dim(X, \tau) \le n$ if and only if there exist topologies $\tau_1, \dots, \tau_{n+1}$ for which $\dim(X, \tau_i) \le 0$ and

$$\tau = \bigcap_{i=1}^{n+1} \tau_i.$$

They set the problem of verifying this hypothesis in various classes of completely regular spaces, in particular in the class of metrizable spaces.

The reader may judge how close the following result is to the solution of the latter problem.

Theorem. Let (X, τ) be a metrizable space. Then $\dim(X, \tau) \le n$ if and only if there exist topologies $\tau_1, \dots, \tau_{n+1}$ such that the following conditions are satisfied:

a) (X, τ_i) is a metrizable space, $i = 1, \dots, n+1$;

b) $(X, \bigcap_{i=1}^{k} \tau_i)$ is a metrizable space for each $k = 1, \dots, n+1$;

c) $\dim(X, \tau_i) \le 0$;

d) $\tau = \bigcap_{i=1}^{n+1} \tau_i$.

Proof. Necessity:

Let $\dim(X,\tau) \le n$. Then there exist such $A_i \subset X$, $i = 1, \dots, n+1$,

$$X = \bigcup_{i=1}^{n+1} A_i,$$

and $\dim(A_i \tau) \le 0$. Moreover, one may assume that the A_i are of type G_δ, $i = 1, \dots, n+1$ (see ([2])).

Denote by τ_i the topology whose base is the union of the family τ and the family of all one-point subsets of the set $X \setminus A_i$. Since τ forms a base of the topology τ_i at the points of the set A_i and $\tau \subset \tau_i$, $i = 1, \dots, n+1$, we conclude that

$$\tau = \bigcap_{i=1}^{n+1} \tau_i.$$

Fix the value of the index i and put $\tau_i = \tau'$, $A_i = A'$.

Lemma. (X, τ') is a normal space.

Proof. We shall prove this without using the fact that there exists a σ-locally finite base in (X, τ') (it is obvious that the space (X, τ') is regular).

Let D and C be disjoint τ'-closed sets; then

$$D = D_1 \setminus D_2, \qquad C = C_1 \setminus C_2,$$

where D_1, C_1 are τ-closed and $D_2, C_2 \subset X \setminus A'$.

$$D \cap C = \Lambda = (D_1 \setminus D_2) \cap (C_1 \setminus C_2) = (D_1 \cap C_1) \setminus (D_2 \cup C_2).$$

It follows from this equality that

$$D_1 \cap C_1 \subset D_2 \cup C_2 \subset X \setminus A'.$$

Thus, $D_1 \cap C_1$ is τ'-open and τ'-closed.

Let

$$L = X \setminus (D_1 \cap C_1).$$

Then $L \cap D_1$, $L \cap C_1$ are disjoint τ-closed sets in L, and, consequently, there exist τ-open sets U_1, V_1 in L such that

$$L \cap D_1 \subset U_1, \qquad L \cap C_1 \subset V_1, \qquad \overline{U}_1 \cap \overline{V}_1 = \Lambda, \qquad U_1 \cup V_1 \subset L.$$

Since L is τ'-open in X, the sets U_1 and V_1 are τ'-open. And because the discrete topology is introduced on the set $D_1 \cap C_1$, the sets U and V are τ'-open and

$$U \supset D, \qquad V \supset C, \qquad U \cap V = \Lambda,$$

where

$$U = U_1 \cup ((D_1 \cap C_1) \cap D), \qquad V = V_1 \cup ((D_1 \cap C_1) \cap C).$$

The lemma is proved.

a) (X, τ') is a metrizable space.

Proof. $X \setminus A'$ has type F_σ in the topology τ:

$$X \setminus A' = \bigcup_{j=1}^{\infty} F_j.$$

Thus a base B' of the topology τ' can be represented in the form

$$B' = B_\tau \cup \left(\bigcup_{j=1}^{\infty} \{\{x\} \mid x \in F_j\} \right),$$

where B_τ is a σ-locally finite base of the topology τ. Clearly, B' is σ-locally finite.

b) $\left(X, \bigcap_{i=1}^{k} \tau_i\right)$ is a metrizable space.

This assertion follows immediately from the definition of the topologies τ_i and property a) (the intersection of a finite family of sets of type F_σ has type F_σ).

c) $\dim(X, \tau') \leq 0$.

Proof. Let $\dim(A', \tau) \leq k$, and let $\{\Gamma_j\}$ be a finite covering of the set X by τ'-open sets. Then in the covering $\{A' \cap \Gamma_j\}$ of the τ'-closed set A' we combinatorially inscribe a τ'-closed covering $\{K_j\}$ of multiplicity $k+1$ (this is possible, since on the set A' the topologies τ and τ' coincide). Since (X, τ') is a normal space, one can construct τ'-open sets $\{V_j\}$ such that

$$K_j \subset V_j \subset \Gamma_j$$

and the multiplicity of the family $\{V_j\}$ does not exceed $k+1$. Let $\{U_j\}$ be an arbitrary covering of multiplicity 1 of the set

$$X \setminus \left(\bigcup_j V_j \right) \subset X \setminus A',$$

combinatorially inscribed in

$$\left\{ \Gamma_j \cap \left(X \setminus \left(\bigcup_j V_j \right) \right) \right\}.$$

Then $\{U_j \cup V_j\}$ is a τ'-open covering of X, combinatorially inscribed in $\{\Gamma_j\}$, of the required multiplicity.

Remark. If one dispenses with property b), then one can ensure that the weight of each (X, τ_i) does not exceed the weight of the space. Indeed (see ([1])), take, for some i, a σ-locally finite base $B_i = \{U_\alpha\}$ such that

$$\mathrm{Fr}(U_\alpha) \cap A_i = \Lambda.$$

It is clear that the family

$$B_i' = \{U_\alpha\} \cup \{\mathrm{Fr}\, U_\alpha\}$$

is σ-locally finite. Moreover, the family B_i'', obtained from B_i' by finite intersections of elements, is also σ-locally finite. Declare B_i'' to be a base of some topology τ_i. From the construction it is seen that B_i'' consists of τ_i-open-closed sets. Hence it is clear that (X, τ_i) is regular, metrizable, and zero-dimensional; moreover, at the points of A_i there exists a base consisting of sets open in the topology τ, i.e.

$$\bigcap_{i=1}^{n+1} \tau_i = \tau.$$

It is clear that the weight of the space has not changed.

Sufficiency. We shall prove a more general assertion: let (X, τ) be a hereditarily normal Fréchet-Urysohn space and

$$\dim(X, \tau) \geq n + 1.$$

Suppose also that topologies $\tau_1, \dots, \tau_{n+1}$ are given on the set X such that:

a′) (X, τ_i) is a hereditarily normal Fréchet-Urysohn space;

b′) $\left(X, \bigcap_{i=1}^{k} \tau_i\right)$ is a hereditarily normal Fréchet-Urysohn space for each $k = 1, \dots, n+1$;

c′) $\dim(X, \tau_i) \leqslant 0$, then

$$\tau \neq \bigcap_{i=1}^{n+1} \tau_i.$$

We shall prove this assertion by induction: for $n = 0$ the assertion is obvious. Suppose that we have proved that $\dim(X, \tau') \leqslant n - 1$, where

$$\tau' = \bigcap_{i=1}^{n} \tau_i.$$

Assume that $\tau' \cap \tau_{n+1} = \tau$.

Lemma 1. *Let $(X, \tau_1), \dots, (X, \tau_k)$ and $(X, \bigcap_{i=1}^{k} \tau_i)$ be Fréchet-Urysohn spaces; then*

$$[A]_{\bigcap_{i=1}^{k} \tau_i} = \bigcup_{i=1}^{k} [A]_{\tau_i},$$

where $A \subset X$ and $[A]_\tau$ denotes the closure of the set in the topology τ.

Proof. Let

$$x \in [A]_{\bigcap_{i=1}^k \tau_i} \quad \text{and} \quad x \notin \bigcup_{i=1}^k [A]_{\tau_i}.$$

Since $(X, \bigcap_{i=1}^k \tau_i)$ is a Fréchet–Urysohn space, there exists a sequence $\{x_n\} \subset A$ such that $\{x_n\}$ converges to x in the topology $\bigcap_{i=1}^k \tau_i$, but for each i the set $\{x_n\}$ is closed in the topology τ_i and, consequently, closed in the topology $\bigcap_{i=1}^k \tau_i$—a contradiction (the reverse inclusion is obvious).

Lemma 2. *Let (X, τ_1), (X, τ_2), and $(X, \tau_1 \cap \tau_2)$ be hereditarily normal Fréchet-Urysohn spaces, where $\dim(X, \tau_1) \leqslant n$ and $\dim(X, \tau_2) \leqslant m$. Then*

$$\dim(X, \tau_1 \cap \tau_2) \leqslant n + m + 1.$$

Proof. Suppose the contrary. Then there exist (see (1)) pairs of $\tau_1 \cap \tau_2$-closed sets

$$\{C_i, C_i'\}, \quad i = 1, \dots, n + m + 2,$$

such that

$$C_i \cap C_i' = \Lambda$$

and for every collection $\{B_i\}$, $i = 1, \dots, n + m + 2$, where B_i is a partition between C_i and C_i',

$$\bigcap_{i=1}^{n+m+2} B_i \neq \Lambda$$

(B_i is τ-closed, where $\tau = \tau_1 \cap \tau_2$).

Since $\dim(X, \tau_1) \leqslant n$, there exist $n+1$ pairs of τ_1-open sets

$$\{V_i, V_i'\}, \quad i = 1, \dots, n + 1,$$

such that

$$[V_i]_{\tau_1} \cap [V_i']_{\tau_1} = X \setminus (V_i \cup V_i'), \qquad V_i \supset [\Gamma_i]_\tau \supset \Gamma_i \supset C_i, \quad V_i' \supset [\Gamma_i']_\tau \supset \Gamma_i' \supset C_i',$$

and $\{V_i, V_i'\}$ is a cover (here Γ_i and Γ_i' are τ-open sets).

Consider also a τ_2-open cover

$$\{U_i, U_i'\}, \quad i = n + 2, \dots, n + m + 2,$$

having analogous properties.

Introduce the notation

$$B_i = [V_i]_\tau \cap [V_i']_\tau, \quad i = 1, \dots, n + 1.$$

It is clear that B_i is a partition between C_i and C_i' in the topology τ.

Assertion. Let

$$D = \bigcap_{i=1}^{n+1} B_i.$$

Then there is a pair

$$U, U' \in \{U_i, U'_i\}, \quad i = n+2, \dots, n+m+2,$$

such that $D \cap U$ contains some point x which is a τ-limit point for $D \cap U'$ (or conversely).

Proof. If this were not so, then there would exist a partition B_i, closed in the topology τ, between

$$C_i \cup (D \cap U_i) \quad \text{and} \quad C'_i \cup (D \cap U'_i)$$

for each $i = n+2, \dots, n+m+2$ (by the assumption: the space (X, τ) is hereditarily normal and C_i does not meet $[V'_i]_\tau$). Note that

$$D \cap B_i \subset X \setminus (U_i \cup U'_i),$$

whence follows the equality

$$\bigcap_{i=n+2}^{n+m+2} (D \cap B_i) = \Lambda,$$

which contradicts the choice of the sets $\{C_i, C'_i\}$.

Thus there exists

$$x \in (D \cap U) \cap [D \cap U']_\tau.$$

Since the family $\{V_i, V'_i\}$ is a cover, we may assume that $x \in V$ for some pair

$$V, V' \in \{V_i, V'_i\}.$$

It follows from Lemma 1 that $U \cap V$ contains no points that are τ-limit points for $U' \cap V'$. Knowing this, we arrive at a contradiction.

Since

$$x \in (U \cap D) \cap [U' \cap D]_\tau,$$

there exists a sequence $\{x_n\} \subset D \cap U'$ converging to x in the topology τ; then, by Lemma 1, $\{x_n\}$ converges to x either in the topology τ_1 or in the topology τ_2, but U is τ_2-open and, consequently, $\{x_n\}$ converges to x in the topology τ_1.

Since $\{x_n\} \subset D \subset [V']_\tau$, for each n there exists a sequence $\{y_n^i\} \subset V'$, converging with respect to i to x_n in the topology τ, and $\{y_n^i\}$ cannot converge to x_n in the topology τ_1 for any subsequence of the indices n cofinal in the natural numbers; otherwise

$$x \in [\bigcup_{i,n} y_n^i]_{\tau_1},$$

whereas we have $x \in V = X \setminus [V']_{\tau_1}$.

Thus $\{y_n^i\}$ converges with respect to i to x_n in the topology τ_2 (at least starting from some n), and therefore it may be assumed that $\{y_n^i\} \subset U' \cap V'$; but since (X, τ) is a Fréchet-Urysohn space, it follows that

$$x \in [\bigcup_{i,n} y_n^i]_\tau,$$

a contradiction (this is true because U' is τ_2-open and $\{x_n\} \subset U'$).

At first glance it seems that property (δ') is superfluous, automatically satisfied. I do not know whether one can dispense with it, but in any case, as the example of A. V. Arhangel' skii shows, it is not automatic.

Example. Consider the set of points in the plane

$$X = (0,0) \cup \left\{ \left(\frac{1}{n}, 0 \right) \right\} \cup \left\{ \left(\frac{1}{n}, \frac{1}{m} \right) \right\},$$

where n, m are natural numbers. In addition to the usual open sets induced by the topology of the plane, declare open:

1) in the topology τ_1, the point $(0,0)$;

2) in the topology τ_2, the set

$$(0,0) \cup \left\{ \left(\frac{1}{n}, 0 \right) \right\}.$$

It is clear that in the topology $\tau_1 \cap \tau_2$ the point $(0,0)$ has no countable base, and the space $(X, \tau_1 \cap \tau_2)$ is not a Fréchet-Urysohn space (from the set

$$\left\{ \left(\frac{1}{n}, \frac{1}{m} \right) \right\}$$

one cannot reach the point $(0,0)$ by any countable sequence, although

$$\left[\left\{ \left(\frac{1}{n}, \frac{1}{m} \right) \right\} \right]_{\tau_1 \cap \tau_2} = X$$

).

Moscow State University
named after M. V. Lomonosov

Received
3 IV 1970

REFERENCES

1. R. Engelking, *Outline of General Topology*, Amsterdam, p. 291, 1968.

2. Yu. M. Smirnov, *Izv. AN SSSR, Ser. Mat.*, 20, No. 2 (1956).

Note: Figure translations are in progress. See original paper for figures.