# ON EVERYWHERE DIVERGENT EXTENDED HERMITE-FEJÉR INTERPOLATION PROCESSES

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## Abstract

## Full Text

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*MATHEMATICS*

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# ON EVERYWHERE DIVERGENT EXTENDED HERMITE-FEJÉR INTERPOLATION PROCESSES

*(Presented by Academician L. V. Kantorovich on 29 XII 1969)*

## 1°. Introduction.

In 1916 L. Fejér ([1]) proved the following important theorem.

Let $f(x)$ be an arbitrary continuous function on the segment $[-1, 1]$, and let $H_n(f, x)$ be the polynomial of degree $(2n-1)$, uniquely determined by the conditions

$$H_n(f, x_k^{(n)}) = f(x_k^{(n)}), \qquad H_n'(f, x_k^{(n)}) = 0, \qquad k = 1, 2, \dots, n,$$

$$x_k^{(n)} = \cos\frac{2k-1}{2n}\,\pi. \tag{1}$$

Then the relation

$$H_n(f, x) \to f(x), \qquad n \to \infty$$

holds uniformly on $[-1, 1]$.

Let us now consider the so-called extended Hermite-Fejér interpolation polynomial $F_n(f, x)$ of degree $(2n+3)$, which is uniquely determined by the equalities

$$F_n(f, 1) = f(1), \qquad F_n(f, -1) = f(-1), \qquad F_n'(f, 1) = 0, \tag{2}$$

$$F_n'(f, -1) = 0, \qquad F_n(f, x_k^{(n)}) = f(x_k^{(n)}), \qquad F_n'(f, x_k^{(n)}) = 0, \qquad k = 1, 2, \dots, n,$$

where $\{x_k^{(n)}\}_{k=1}^n$ are the Chebyshev nodes (1). In ([2], [3]) it was proved that the process constructed even for such a simple function as $|x|$ diverges at $x = 0$. Since the node matrix of the process $\{F_n\}_{n=1}^\infty$ is obtained by extending the node matrix of the process $\{H_n\}_{n=1}^\infty$ by adding the points $\pm 1$ as nodes, this result, in view of the aforementioned theorem of L. Fejér, is unexpected. We now weaken conditions (2); namely, we consider the polynomial $A_n(f)$ of degree $(2n+2)$, which is uniquely determined by the equalities

$$A_n(f,1) = f(1), \qquad A_n(f,-1) = f(-1), \qquad A_n'(f,1) = 0,$$

$$A_n(f, x_k^{(n)}) = f(x_k^{(n)}), \tag{3}$$

$$A_n'(f, x_k^{(n)}) = 0, \qquad k = 1, 2, \dots, n,$$

where $x_k^{(n)}$ are, as before, the nodes (1). The polynomials $A_n$ are in a certain sense closer to the polynomials $H_n$ than are the polynomials $F_n$. Therefore the question of whether the process $\{A_n(f)\}_{n=1}^\infty$ will converge uniformly for every function continuous on $[-1,1]$ is of definite interest. Recently R. B. Saxena ([4]), with the aid of results from ([3]), proved simply that the process $\{A_n(f)\}_{n=1}^\infty$ diverges at $x = 0$ if $f(x) = |x|$. Thus here the same situation occurs as in the case of the process $\{F_n(f)\}_{n=1}^\infty$.

In connection with the above, the following questions arise:

1. Does there exist a function continuous on $[-1,1]$ for which the process $\{A_n(f,x)\}_{n=1}^\infty$ diverges at all points of $(-1,1)$?
2. Does there exist a function continuous on $[-1,1]$ for which the process $\{F_n(f,x)\}_{n=1}^\infty$ diverges at all points of $(-1,1)$?

In this note only the first question is considered. The second question is resolved in an analogous way.

**2°. Formulation and proof of the theorem.**

**Theorem 1.** *The interpolation process $\{A_n(f,x)\}_{n=1}^\infty$, constructed at the nodes (1) for $f(x) = 1 - x^2$, diverges at all points of the interval $(-1,1)$.*

**Proof.** We need the following

**Lemma.** *For any $\theta \in [0, \pi/2]$ one can find a sequence of natural numbers $n_1 < n_2 < \cdots,\ n_k \to \infty,\ k \to \infty$, such that the equality*

$$\lim_{k\to\infty} \sin^2 n_k\theta = 0$$

*holds.*

The polynomial $A_n(f,x)$, uniquely determined by the conditions (2), has the form

$$A_n(f,x) = \left[\left(\frac{1-x}{2}\right)^2 f(-1) + \frac{1+x}{2}\left\{1 + \frac{4n^2+1}{2}(1-x)\right\} f(1)\right] T_n^2(x) +$$
$$+ \sum_{\nu=1}^{n} f(x_\nu^{(n)}) \frac{(1-x^2)(1-x)}{(1-x_\nu^2)(1-x_\nu)} \frac{(1-x_\nu^2)+(1+2x_\nu)(x-x_\nu)}{n^2(x-x_\nu)^2} T_n^2(x),$$
$$T_n(x) = \cos n \arccos x. \tag{4}$$

Therefore, for $f(x) = 1-x^2$ we have:

$$A_n(f,x) = \frac{(1-x^2)(1-x)T_n^2(x)}{n^2}\left[\sum_{\nu=1}^{n} \frac{1+x_\nu}{(x-x_\nu)^2} + \sum_{\nu=1}^{n} \frac{1+2x_\nu}{(1-x_\nu)(x-x_\nu)}\right]. \tag{5}$$

Let us note that

$$\frac{1+2x_\nu}{(1-x_\nu)(x-x_\nu)} = \frac{1+2x}{(1-x)(x-x_\nu)} + \frac{3}{(x-1)(1-x_\nu)}.$$

Consequently, (5) can be written in the form

$$A_n(f,x) = \frac{(1-x^2)(1-x)T_n^2(x)}{n^2}\left[\sum_{\nu=1}^{n} \frac{1}{(x-x_\nu)^2} + \sum_{\nu=1}^{n} \frac{x}{(x-x_\nu)^2} - \sum_{\nu=1}^{n} \frac{1}{x-x_\nu}\right.$$
$$\left. + \frac{1+2x}{1-x}\sum_{\nu=1}^{n} \frac{1}{x-x_\nu} + \frac{3}{x-1}\sum_{\nu=1}^{n} \frac{1}{1-x_\nu}\right]. \tag{6}$$

It is known that

$$\frac{T_n'(x)}{T_n(x)} = \sum_{\nu=1}^{n} \frac{1}{x-x_\nu}, \qquad \frac{(1-x^2)T_n^2(x)}{n^2}\sum_{\nu=1}^{n} \frac{1}{(x-x_\nu)^2} = 1 - \frac{\sin 2n\theta \cos\theta}{2n\sin\theta}, \quad x = \cos\theta,$$
$$\sum_{\nu=1}^{n} \frac{1}{1-x_\nu} = n^2. \tag{7}$$

Therefore from (6) we derive

$$A_n(f,x) = (1-x^2) + \frac{\sin 2n\theta \sin 2\theta}{2n} - 3\sin^2\theta\cos^2 n\theta, \qquad x = \cos\theta. \tag{8}$$

Consider first the case when $0 \le x < 1$. Suppose that, for some $x$ in $[0,1)$, convergence of the process $\{A_n(f,x)\}_{n=1}^{\infty}$ takes place. Then, by (8), we have

$$\sin 2n\theta \, \sin 2\theta/2n - 3\sin^2\theta\cos^2 n\theta \to 0, \qquad n \to \infty. \tag{9}$$

This equality is equivalent to the equality

$$\lim_{n\to\infty} \cos^2 n\theta = 0, \qquad \theta \in (0, \pi/2],$$

which contradicts the lemma. Consequently, in $[0, 1)$ the process diverges.

Consider now the interval $(-1, 0)$. If at some point $\tilde{x} \in (-1, 0)$ the process $\{A_n\}_{n=1}^{\infty}$ converged, then, according to the preceding, $\lim_{n\to\infty} \cos^2 n\bar{\theta} = 0$ (9), where $\tilde{x} = \cos\bar{\theta}$. Put $\theta = \pi - \bar{\theta}$, $\pi/2 < \bar{\theta} < \pi$. Thus $0 < \theta < \pi/2$. Since

$$\cos^2 n\bar{\theta} = \cos^2 n\theta,$$

it follows from (9) that $\lim_{n\to\infty} \cos n\theta = 0$. This again contradicts the lemma. Thus the process $\{A_n(f)\}_{n=1}^{\infty}$ also diverges on the interval $(-1, 0)$. The theorem is proved.

Along with the polynomial $A_n(f, x)$, consider the polynomial $B_n(f, x)$, uniquely determined by the conditions

$$B_n(f, 1) = f(1), \qquad B_n(f, -1) = f(-1), \qquad B_n'(f, -1) = 0,$$

$$B_n(f, x_k^{(n)}) = f(x_k^{(n)}), \qquad B_n'(f, x_k^{(n)}) = 0, \qquad k = 1, 2, \dots, n.$$

By an almost verbatim repetition of the arguments in the proof of Theorem 1, we obtain Theorem 2.

**Theorem 2.** *The interpolation process $\{B_n(f, x)\}_{n=1}^{\infty}$, constructed at the Chebyshev nodes (1) for $f(x) = 1 - x^2$, diverges at all points of the interval $(-1, 1)$.*

**Corollary.** *The interpolation process $\{A_n(f)\}_{n=1}^{\infty}$, constructed at the Chebyshev nodes (1) for $f(x) = x^2$, diverges at all points of the interval $(-1, 1)$.*

Indeed, for $f \equiv 1$, $A_n(f) \equiv 1$, hence

$$A_n(1 - z^2, x) = 1 - A_n(z^2, x). \tag{10}$$

If, for $x \in (-1, 1)$, $A_n(z^2, x) \to x^2$, $n \to \infty$, then, according to (10), $A_n(1 - z^2, x) \to 1 - x^2$, $n \to \infty$, which contradicts Theorem 1.

**Theorem 3.** *The interpolation process $\{A_n(f)\}_{n=1}^{\infty}$, constructed at the Chebyshev nodes (1) for $f(x) = x$, diverges at all points of the interval $(-1, 1)$.*

**Proof.** Since $A_n(1, x) \equiv 1$, according to (4),

$$1 - A_n(z, x) = T_n^2(x) \left[ \frac{(x-1)^2}{2} + \frac{(1-x)(1-x^2)}{n^2} \sum_{\nu=1}^{n} \frac{(1-x_\nu^2) + (1+2x_\nu)(x-x_\nu)}{(1-x_\nu^2)(x-x_\nu)^2} \right].$$

Hence, after simple transformations, we obtain

$$1 - A_n(z,x) = T_n^2(x)\left[\frac{(x-1)^2}{2} + \frac{(1-x)(1-x^2)}{n^2}\left(\frac{3}{2(x-1)}\sum_{\nu=1}^{n}\frac{1}{1-x_\nu} - \sum_{\nu=1}^{n}\frac{1}{2(x+1)(1+x_\nu)} + \frac{2x+1}{1-x^2}\sum_{\nu=1}^{n}\frac{1}{x-x_\nu} + \sum_{\nu=1}^{n}\frac{1}{(x-x_\nu)^2}\right)\right]. \tag{11}$$

Expression (11), after application of identity (7) and the identity

$$\sum_{\nu=1}^{n} 1/(1+x_\nu) = n^2$$

takes the form

$$1 - A_n(z,x) = (1-x) - \frac{(1-x)\sin 2n\theta\cos\theta}{2n\sin\theta} + \frac{3(x^2+1)\cos^2 n\theta}{2} + \frac{(2x+1)(1-x)\cos n\theta\sin n\theta}{n\sin\theta}, \qquad x = \cos\theta. \tag{12}$$

If $A_n(z,x) \to x$, $n \to \infty$, then from (12) it would follow that $\lim_{n\to\infty}\cos^2 n\theta = 0$, and this contradicts the lemma.

**Remark.** In connection with Theorems 1 and 3 it is curious that if the process $\{A_n(f)\}_{n=1}^{\infty}$ is constructed for $f(x) = (1-x^2)(1-x)$, then it converges uniformly. Moreover,

$$|A_n(f) - f| \le 12/n, \qquad |x| \le 1, \qquad f(x) = (1-x^2)(1-x). \tag{13}$$

Indeed, in the present case

$$A_n(f,x) = \frac{(1-x)(1-x^2)T_n^2(x)}{n^2}\sum_{\nu=1}^{n}\left(\frac{1-x^2}{(x-x_\nu)^2} - \frac{4x+1}{x-x_\nu} - 3\right).$$

Hence, with the aid of the identities (7), we obtain

$$A_n(f,x) = (1-x)(1-x^2) - \frac{(1-x)\sin 2n\theta\sin 2\theta}{4n} - \frac{(4x+1)(1-x)\sin 2n\theta\sin\theta}{2n} - \frac{3(1-x)\cos^2 n\theta\sin^2\theta}{n}.$$

Thus, (13) holds.

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*Note: Figure translations are in progress. See original paper for figures.*