ESTIMATES IN THE NORMS $H_{(s)}$ AND HYPOELLIPTICITY

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Abstract

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ESTIMATES IN THE NORMS $H_{(s)}$ AND HYPOELLIPTICITY

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Let P be a linear differential operator with infinitely differentiable coefficients, defined in a domain $\Omega \subseteq \mathbf{R}^n$ (or on a manifold of class C^∞). The operator P is called hypoelliptic if, for every distribution u in Ω, one has

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} Pu.$$

In this paper we give sufficient conditions for hypoellipticity that are more general than those in [1]. With their help it is possible to prove the hypoellipticity of certain second-order differential operators with real coefficients and a semidefinite principal part which do not satisfy the conditions obtained in [4, 6].

For convenience of formulation, let us introduce the conditions A, B, and C written below. As usual, $[P, l]$ denotes the commutator of the operators P and l.

A. For every point $x_0 \in \Omega$ there exist a neighborhood $U(x_0)$ and real numbers s_0 and a such that, whatever the real number N, with some constant C_N independent of u, the inequality

$$\|u\|^2_{(s_0+a)} \le C_N\{\|Pu\|^2_{(s_0)} + \|u\|^2_{(-N)}\}, \qquad u \in C_0^\infty(U(x_0)). \tag{1}$$

holds.

Since in inequality (1), for all x belonging to some compact set K, one may take one and the same a, we shall assume that a does not depend on x.

B. For every point $x_0 \in \Omega$ there exist a neighborhood $U(x_0)$ and a real number s_1 such that, whatever the linear differential operator l of order zero or one with infinitely differentiable coefficients in Ω, and whatever the real $s > s_1$, $\varepsilon > 0$, and N, with some constant $C_{\varepsilon,l,N,s}$ independent of u, the inequality

$$\|[P, l]u\|^2_{(s)} \le \varepsilon\|Pu\|^2_{(s+\operatorname{ord} l)} + C_{\varepsilon,l,N,s}\|u\|^2_{(-N)}, \qquad u \in C_0^\infty(U(x_0)). \tag{2}$$

holds.

C. For every point $x_0 \in \Omega$ and every neighborhood $U(x_0)$ there exist a real number s_2 and a sequence of functions $\{\psi_j(x)\}$ $(\psi_j(x) \in C_0^\infty(U(x_0)),\ \psi_1(x_0) \neq 0,\ \psi_{j+1}$ is equal to one in a neighborhood of the support of $\psi_j)$ such that, for arbitrary real $s > s_2$ and N, with some $\varkappa > 0$ and constants $C_{j,N,s}$ independent of u, the inequalities

$$\|[P, \psi_j]u\|^2_{(s)} \leq C_{j,N,s}\{\|Pu\|^2_{(s-\varkappa)} + \|u\|^2_{(-N)}\}, \qquad u \in C_0^\infty(U(x_0)). \tag{3}$$

hold.

Theorem 1. *If the operator P satisfies conditions A and B in Ω, then from $u \in \mathcal{E}'(\Omega)$, $Pu \in H_{(s)}(\Omega)$, it follows that $u \in H_{(s+a)}(\Omega)$.*

Consider the operator P, equal to

$$\sum_{j=1}^{r} X_j^2 + X_0 + c,$$

where all X_j are homogeneous first-order differential operators with real infinitely differentiable coefficients in Ω.

Definition. We shall say that the operator P satisfies Hörmander' s condition at the point $x \in \Omega$ if at this point the Lie algebra generated by the operators $X_0, \dots, X_r$ has rank n. In [4] it is proved that,

that if P satisfies this condition at every point of Ω, then it is hypoelliptic.

The following lemma gives an example of operators satisfying the conditions of Theorem 1.

Lemma. Let M be a smooth submanifold in Ω of dimension $\leq n-1$. Suppose that at all points $x \in \Omega \setminus M$ the operator

$$P = \sum_{j=1}^{r} X_j^2 + X_0 + c$$

satisfies the Hör condition, while at those points $x \in M$ where this condition is violated, at least one of the operators X_j $(j = 1, \dots, r)$ is transversal to M. Then the operator P satisfies condition A with $a = 0$ and condition B.

If Ω is a domain in $\mathbb{R}^n$, then condition B is equivalent to condition B′, which is more convenient for use.

B′. For any point $x_0 \in \Omega$ there exists a neighborhood $U(x_0)$ such that for any multi-indices α and β, $|\alpha| + |\beta| \neq 0$, and certain real s_β^α, whatever $\varepsilon > 0$ and N may be, the inequalities

$$\left\|P_{(\beta)}^{(\alpha)} u\right\|^2_{(s_\beta^\alpha)} \leq \varepsilon \|Pu\|^2_{(s_\beta^\alpha + |\beta|)} + C\|u\|^2_{(-N)}, \qquad u \in C_0^\infty(U(x_0)). \tag{4}$$

hold with certain constants independent of u, but possibly depending on $x_0, \alpha, \beta, \varepsilon$, and N.

Here $P^{(\alpha)}_{(\beta)}$ is the differential operator corresponding to the symbol $D^{\alpha}_{\xi} D^{\beta}_{x} p(x, \xi)$, where $p(x, \xi)$ is the symbol of the operator P.

Remark. It can be shown that from the validity of B′ it follows that the inequalities (4) hold with arbitrary s in place of s^{α}_{β} in the exponents of the norms. Conversely, if the inequalities (4) hold only for $|\alpha| + |\beta| = 1$, but for all s, then condition B′ is fulfilled. From the validity of condition B′ there follows the validity of condition B″.

B″. For any compact $K \subset \Omega$, any multi-indices β, $|\beta| \neq 0$, and certain real s_β, whatever $\varepsilon > 0$ and N may be, the inequalities

$$\left\| P_{(\beta)} u \right\|^2_{(s_\beta)} \leq \varepsilon \| Pu \|^2_{(s_\beta + |\beta|)} + C \| u \|^2_{(-N)}, \qquad u \in C^{\infty}_0(K). \tag{5}$$

hold with certain constants independent of u.

Theorem 1′. Let the operator P satisfy conditions A and B″ in some domain $\Omega \subseteq \mathbb{R}^n$. Then from $u \in D'(\Omega)$, $Pu \in H^{\mathrm{loc}}_{(s)}(\Omega)$, and $\operatorname{sing\,supp} u$ being compact in Ω, it follows that $u \in H^{\mathrm{loc}}_{(s+a)}(\Omega)$.

Let us note that conditions A and B″ are satisfied for all differential operators with constant coefficients. Therefore Theorem 1′ may be regarded as a certain generalization of a theorem of M. S. Agranovich ([7]).

The following theorem is the main result of the paper.

Theorem 2. If the operator P satisfies conditions A, B, and C in Ω, then from $u \in D'(\Omega)$, $Pu \in H^{\mathrm{loc}}_{(s)}(\Omega')$, where Ω' is an arbitrary open subset in Ω, it follows that $u \in H^{\mathrm{loc}}_{(s+a)}(\Omega')$. Consequently, the operator P is hypoelliptic.

In the proof of Theorems 1 and 2, mollifiers studied in ([2,3]) are used.

Theorem 3. If the operator

$$P = \sum_{j=1}^{r} X_j^2 + X_0 + c$$

at every point of Ω, except for a certain set Q of isolated points, satisfies the Hör condition, and at every point $x \in Q$ at least one of the operators X_j $(j = 0, 1, \ldots, r)$ is nonzero, then from $u \in D'(\Omega)$, $Pu \in H^{\mathrm{loc}}_{(s)}(\Omega')$, it follows that $u \in H^{\mathrm{loc}}_{(s)}(\Omega')$. Consequently, the operator P is hypoelliptic.

Theorem 4. Consider the differential operator $P = -\partial^2/\partial x^2 - \varphi^2(x)\partial^2/\partial y^2$. Suppose that $\varphi(x)$ belongs to $C^{\infty}(\mathbf{R}^1)$, is equal to zero at $x = 0$ together with all its derivatives, and is different from zero for $x \neq 0$. Then from $u \in D'(\Omega)$, $Pu \in H^{\mathrm{loc}}_{(s)}(\Omega')$, it follows that $u \in H^{\mathrm{loc}}_{(s)}(\Omega')$. Consequently, the operator P is hypoelliptic.

Theorems 3 and 4 are proved by using Theorem 2. The greatest difficulty is the verification that the operator $-\partial^2/\partial x^2 - \varphi^2(x)\partial^2/\partial y^2$ satisfies condition C. By Fourier transformation with respect to y, this problem is reduced to proving the estimate

$$\left\| |\eta|^{\varkappa} \frac{d}{dx}\varphi^2 u \right\|^2_{(0)} + \left\| |\eta|^{1+\varkappa}\varphi^2 u \right\|^2_{(0)} \leq C\left\{ \left\| Q_\eta u \right\|^2_{(0)} + \|u\|^2_{(0)} \right\}, \qquad u \in C_0^\infty(K). \tag{6}$$

Here K is some compact set in $\mathbf{R}^1$, $Q_\eta = -d^2/dx^2 + \varphi^2(x)\eta^2$, the constant C does not depend on η, and $\varkappa$ is some real number greater than zero. First, estimate (6) is proved when the support of u belongs to the set $\{x : \ |\varphi(x)| > b/\sqrt{|\eta|}\}$, where b is a certain constant depending on the function $\varphi(x)$. In doing so, the method of paper (^{5}) is used. Then estimate (6) is proved by integration by parts when the support of u belongs to the set $\{x : \ |\varphi(x)| < 2b/\sqrt{|\eta|}\}$. The proof is completed by using a certain special partition of unity.

Remark. Theorems 1 and 2 are also valid for pseudodifferential operators with compact supports.

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