ON THE STABILITY OF THE DIRICHLET AND NEUMANN PROBLEMS

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Abstract

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MATHEMATICS

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ON THE STABILITY OF THE DIRICHLET AND NEUMANN PROBLEMS

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Let $C(F)$ be the capacity ([1]) of a bounded closed set F in three-dimensional Euclidean space. Following the concept of analytic capacity (([2]), $p.103$), we shall call the capacity $C(E)$ of an arbitrary set E in this space the number

$$C(E) = \sup_{F \subset E} C(F).$$

We shall say that an open set E has stable capacity if $C(\overline{E}) = C(E)$.

Theorem 1. *A domain E may have stable capacity despite the presence of irregular ([1]) points of the complement to its closure.*

The proof consists in constructing an example. In doing so, the construction of Lebesgue' s well-known example of the non-solvability of the Dirichlet problem (([3]), $p.203$) is used.

Let E be a bounded simply connected domain and $x \in E$. Denote by $\widehat{E}_x$ the complement of the inversion of the domain $\overline{E}$ with respect to the sphere with center at x and fixed sufficiently large radius R.

Theorem 2 (main). *Let the boundary of the domain E have no irregular points and let the complement of $\overline{E}$ be connected. In order that the Dirichlet problem be stable in the closed domain $\overline{E}$ ([1]), it is necessary and sufficient that, for every $x \in E$, the capacity of the domain $\widehat{E}_x$ be stable.*

Remark. As was shown in ([1]), regularity of the boundary alone is insufficient for the stability of the Dirichlet problem. From this example there now follows the existence in three-dimensional space of a Jordan rectifiable domain with regular boundary, but with unstable capacity.

Other criteria for the stability of the Dirichlet problem can be found in ([1,4]). The proof of Theorem 2 is based on the following lemmas and theorems.

Lemma 1. *For every $x \in E$, the potential ([1]) $W_x(y)$ of the closure of the domain $\widehat{E}_x$ and the regular part $g_E(x, y)$ of the Green function of the domain E*

are related by

$$g_E(x,y) = \frac{1}{|x-y|}\, W_x\left(x + R^2 \frac{y-x}{|y-x|^2}\right), \qquad y \in E,$$

$$W_x(y) = \frac{R^2}{|x-y|}\, g_E\left(x, x + R^2 \frac{y-x}{|y-x|^2}\right), \qquad y \in \widehat{\overline{E}}_x. \tag{1}$$

Lemma 2. *Let $\{E_r\}$ be an increasing sequence of domains converging to E in the sense that $\overline{E}_r \subset E$, but $\bigcup E_r = E$. In order that the potentials of the sets $\overline{E}_r$ converge to the potential of the set $\overline{E}$ uniformly on every closed set outside $\overline{E}$, it is necessary and sufficient that*

$$C(\overline{E}) = C(E).$$

Denote by $HL_2'(E)$ the Hilbert space of classes of functions $u(x), v(x)$, $x \in E$, summable over $\overline{E}$, harmonic in E, with scalar product

$$\langle u, v\rangle_E = \frac{1}{4\pi}\int_E (\nabla u \cdot \nabla v)\, dE,$$

where $\nabla u = \operatorname{grad} u$, and the integration is carried out with respect to Lebesgue measure. All functions differing only by a constant term belong to the same class.

Lemma 3. Let the complement of $\bar{E}$ be connected and let $\{E_\rho\}$ be a decreasing sequence of domains converging to E in the sense that $E_\rho \supset \bar{E}$, but $\bigcap E_\rho = \bar{E}$.

In order that the functions $g_{E_\rho}(x,y)$, as functions of $y \in E$, converge weakly in the space $HL_2'(E)$ to the function $g_E(x,y)$, it is necessary and sufficient that the capacity of the domain $\widetilde{E}_x$ be stable.

Proof. The family of functions $g_{E_\rho}(x,y)$ is weakly compact in $HL_2'(E)$ by the well-known Dirichlet principle. Its weak convergence to the function $g_E(x,y)$ now follows from Lemmas 1 and 2. This proves sufficiency. The necessity of the condition is easily established with the aid of the equality

$$C(\widetilde{E}_x) = R^2 g_E(x,x), \tag{2}$$

which follows from (1).

Let $f(x)$ be an everywhere infinitely smooth function. We define the solution u_f of the Dirichlet problem in the space $HL_2'(E)$ with boundary condition $f(x)$ by the equality

$$\langle u, f\rangle_E = \langle u, u_f\rangle_E, \tag{3}$$

where $u \in HL_2'(E)$ is an arbitrary element.

The basis for such a definition is given by the following

Lemma 4. If E is a sufficiently smooth domain, then u_f, up to an additive constant, coincides with the classical solution of the Dirichlet problem.

Proof. The operation

$$A_E f(x) \equiv f(x) + \langle f(y), G_E(x,y)\rangle_E = f(x) - \frac{1}{4\pi}\int_E \Delta f(y)\cdot G_E(x,y)\,dE, \quad (4)$$

where $G_E(x,y)$ is the Green function of the domain E, and Δ is the Laplace operator, which gives the classical solution in the case of a sufficiently smooth domain, projects $f(x)$ into the space $HL_2'(E)$. But u_f, defined by equality (3), is also the projection of $f(x)$ onto $HL_2'(E)$. Consequently, u_f and $A_E f(x)$ are equal as elements of the space $HL_2'(E)$.

Remark. It can be proved that the equality $u_f = A_E f$ is valid if all the domains $\widetilde{E}_x$ have stable capacity.

Let E and $\{E_\rho\}$ satisfy the conditions of Lemma 3, and let $u_{f,\rho} \in HL_2'(E)$ be the solution of the Dirichlet problem in the domain E_ρ.

If u_f is the weak limit of $u_{f,\rho}$ in the space $HL_2'(E)$ as $E_\rho \to E$ for every infinitely smooth function $f(x)$, then we shall call the Dirichlet problem weakly stable in the domain E in the sense of $HL_2'(E)$.

Theorem 3. Let the complement of E be connected. For the weak stability of the Dirichlet problem in the domain E in the sense of $HL_2'(E)$, it is necessary and sufficient that, for every $x \in E$, the domain $\widetilde{E}_x$ have stable capacity.

Proof. Necessity is a consequence of Lemma 3. To prove sufficiency, define the operation A_{E_ρ} by equality (4), with E replaced by E_ρ, and consider the difference

$$A_{E_\rho} f(x) - A_E f(x) = \langle f(y), G_{E_\rho}(x,y)\rangle_{E_\rho \setminus E} + \langle f(y), [g_E(x,y) - g_{E_\rho}(x,y)]\rangle_E,$$
$$x \in E.$$

Hence, by Lemma 3, we have

$$\lim_{E_\rho \to E} A_{E_\rho} f(x) = A_E f(x), \qquad x \in E.$$

But the sequence $A_{E_\rho} f(x)$ is weakly compact in $HL_2'(E)$ and, therefore, converges weakly to $A_E f(x)$ when $E_\rho \to E$.

Theorem 4. Let the complement of $\bar{E}$ be connected. For the stability of the Dirichlet problem inside the domain E (see the definition in $(*)$) it is necessary and sufficient that, for every $x \in E$, the domain $\widetilde{E}_x$ have stable capacity.

Proof. Necessity is established by means of equality (2), sufficiency by means of Theorem 3.

Theorem 2 now follows from Theorem 4 and one of the results of [1].

The space $HL_2'(E)$ has a reproducing function $P_E(x, y, x_0)$, defined by the relation

$$u(x) = u(x_0) + \langle u(y), P_E(x, y, x_0)\rangle_E, \tag{5}$$

where u is an arbitrary element of $HL_2'(E)$, $x, x_0 \in E$, $P_E(x_0, y, x_0) = 0$. The function $P_E(x, y, x_0)$ has the symmetry property $P_E(x, y, x_0) = P_E(y, x, x_0)$.

Theorem 5. *Let E and $\{E_\rho\}$ satisfy the conditions of Lemma 3. In order that, for any $x \in E$, the functions $P_{E_\rho}(x, y, x_0)$, as functions of $y \in E$, converge in the space $HL_2'(E)$ to the function $P_E(x, y, x_0)$, it is necessary and sufficient that the capacities of the domains $\widehat{E}_x$ be stable.*

Proof. It is established that the convergence of the reproducing functions $P_{E_\rho}(x, y, x_0)$ to $P_E(x, y, x_0)$ is equivalent to the completeness of harmonic polynomials in the space $HL_2'(E)$. Next it is proved that the latter, in turn, is equivalent to the weak convergence of the functions $g_{E_\rho}(x, y)$ to $g_E(x, y)$, when $E_\rho \to E$. In doing so one uses the relation

$$\left\langle u(y), \frac{1}{|x-y|}\right\rangle_E = \langle u(y), g_E(x, y)\rangle_E, \qquad u \in HL_2'(E).$$

The proof is completed by reference to Lemma 3.

Let $\varphi(u)$ be a linear functional on the space $HL_2'(E)$. We define the solution u^φ of the Neumann problem in the domain E with boundary condition φ by means of the equality

$$\varphi(u) = \langle u, u^\varphi\rangle_E,$$

where u is an arbitrary element of $HL_2'(E)$.

This formulation of the Neumann problem is a natural generalization of the classical one (cf. the definition of the Neumann problem in [5]).

Let E and $\{E_\rho\}$ satisfy the conditions of Lemma 3, and let $u_\rho^\varphi \in HL_2'(E_\rho)$ be the solution of the Neumann problem in the domain E_ρ with boundary condition φ. If, for any linear functional φ, the elements u_ρ^φ converge weakly in the space $HL_2'(E)$ to the element u^φ, then the Neumann problem will be called stable in the domain E.

Theorem 6. *Let the complement of $\overline{E}$ be connected. For stability of the Neumann problem in the domain E it is necessary and sufficient that the domains $\widehat{E}_x$, $x \in E$, have stable capacity.*

Proof. In view of (5) we have

$$\varphi(u) = \varphi\langle u(y), P_E(x, y, x_0)\rangle_E = \langle u(y), \varphi(P_E(x, y, x_0))\rangle_E.$$

Consequently,

$$u^{\varphi}(y) = \varphi(P_E(x, y, x_0)), \qquad u^{\varphi}_{\rho}(y) = \varphi(P_{E_\rho}(x, y, x_0)). \tag{6}$$

Therefore, by Theorem 5 and the symmetry of the function $P_E(x, y, x_0)$, the elements $u^{\varphi}_{\rho}(y)$ converge pointwise to $u^{\varphi}(y)$ in the domain E, when $E_\rho \to E$. To prove sufficiency it remains to observe that, in view of the boundedness of the functional φ, the family $u^{\varphi}_{\rho}(y)$ is weakly compact in the space $HL'_2(E)$. Necessity of the condition is established with the aid of formulas (6) and Theorem 5.

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Note: Figure translations are in progress. See original paper for figures.