ON SYMBOLS OF ONE-DIMENSIONAL SINGULAR INTEGRAL OPERATORS ON AN OPEN CONTOUR

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ON SYMBOLS OF ONE-DIMENSIONAL SINGULAR INTEGRAL OPERATORS ON AN OPEN CONTOUR

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1. A one-dimensional singular integral operator is an operator A defined by the equality

$$(A\varphi)(t) = c(t)\varphi(t) + \frac{d(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau \qquad (t \in \Gamma), \tag{1}$$

where Γ is a contour in the complex plane; $c(t)$ and $d(t)$ $(t \in \Gamma)$ are prescribed functions (or matrix-functions), which are called the coefficients of the operator A.

In what follows it will be assumed that the contour Γ consists of a finite number of closed and open simple contours of Lyapunov type. Singular integral operators will be considered in the Hilbert space $L_2(\Gamma)$, and in the case where $c(t)$ and $d(t)$ are matrix-functions of order n, in the space $L_2^n(\Gamma)$ of vector-functions $\varphi = \{\varphi_j\}_{j=1}^n$, where $\varphi_j \in L_2(\Gamma)$.

Denote by $\mathfrak{A}_n$ the smallest subalgebra of the Banach algebra $\mathfrak{R}_n$ of all linear bounded operators acting in the space $L_2^n(\Gamma)$, containing all operators of the form (1) with piecewise-continuous coefficients.

In the present note it is proved that the factor algebra $\mathfrak{A}_n/\mathfrak{S}_\infty$, where $\mathfrak{S}_\infty$ is the ideal of all completely continuous operators in $\mathfrak{R}_n$, is isomorphic and isometric to a certain algebra of matrix-functions of order $2n$, defined on the cylinder $\mathfrak{M} = \{(t,\mu) : t \in \Gamma,\ 0 \le \mu \le 1\}$. This isomorphism assigns to each operator in $\mathfrak{A}_n$ its symbol. It is also proved that an operator $A \in \mathfrak{A}_n$ is a Fredholm operator if and only if its symbol is a nonsingular matrix at every point of the cylinder. A formula is established for computing the index of Fredholm operators belonging to the algebra $\mathfrak{A}_n$. For the case when the contour Γ consists only of closed lines, these results were obtained by the authors in paper (^{1}).

2. We shall agree on the following notation: $\Lambda = \Lambda(\Gamma)$ is the set of all functions piecewise-continuous on Γ which are continuous at the ends of the open arcs of the contour Γ and left-continuous on the whole contour Γ; Λ_n is the set of all matrix-functions of order n with elements from Λ; S is the operator of singular integration along Γ:

$$(S\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau \qquad (t \in \Gamma),$$

acting in the space $L_2(\Gamma)$, and S_n is the operator of singular integration in $L_2^n(\Gamma)$, i.e. $S_n\{\varphi_j\}_{j=1}^n = \{S\varphi_j\}_{j=1}^n$. The operator $A = c(t)I + d(t)S$, where $c(t), d(t) \in \Lambda_n$, will be conveniently written in the form

$$A = F(t)P + G(t)Q, \tag{2}$$

where $F(t) = c(t) + d(t)$, $G(t) = c(t) - d(t)$, $P = (I + S_n)/2$, and $Q = (I - S_n)/2$.

Suppose the contour Γ, along with closed lines, contains N open arcs, whose beginnings and ends we denote respectively by α_k and β_k $(k = 1, \dots, N)$.

By the **symbol** of the operator A, defined by equality (2), we shall mean the matrix-function $\mathcal{A}(t, \mu)$ of order $2n$, defined on the cylinder $\mathfrak{M}$ by the equality

$$\mathcal{A}(t,\mu) = \begin{cases} \begin{pmatrix} F(\alpha_k)\mu + G(\alpha_k)(1-\mu) & 0 \\ 0 & F(\alpha_k)\mu + G(\alpha_k)(1-\mu) \end{pmatrix}, & \text{for } t = \alpha_k, \\ \begin{pmatrix} F(t+0)\mu + F(t)(1-\mu) & \sqrt{\mu(1-\mu)}\,(G(t+0) - G(t)) \\ \sqrt{\mu(1-\mu)}\,(F(t+0) - F(t)) & G(t+0)(1-\mu) + G(t)\mu \end{pmatrix}, & \text{for } t \in \Gamma, \quad t = \alpha_k, \beta_k, \\ \begin{pmatrix} F(\beta_k)(1-\mu) + G(\beta_k)\mu & 0 \\ 0 & F(\beta_k)(1-\mu) + G(\beta_k)\mu \end{pmatrix}, & \text{for } t = \beta_k. \end{cases}$$

Theorem 1. In order that the operator $A = F(t)P + G(t)Q$, where $F(t)$, $G(t) \in \Lambda_n$, be a Φ-operator in $L_2^n(\Gamma)$, it is necessary and sufficient that its symbol be nowhere degenerate, i.e.

$$\det \mathcal{A}(t,\mu) \neq 0 \qquad (t \in \Gamma,\ 0 \le \mu \le 1). \tag{3}$$

Proof. Complete the contour Γ to a contour $\tilde{\Gamma}$, consisting of a finite number of closed simple oriented Lyapunov-type curves. Let $\widetilde{F}(t)$ and $\widetilde{G}(t)$ $(t \in \tilde{\Gamma})$ be certain matrix-functions coinciding respectively with $F(t)$ and $G(t)$ on Γ, continuous on the closed set $\overline{\tilde{\Gamma} \setminus \Gamma}$, and nondegenerate at every point of the open set $\tilde{\Gamma} \setminus \Gamma$. By $\widetilde{S}_n$ denote the matrix operator of singular integration along $\tilde{\Gamma}$, and by $\widetilde{P}$ and $\widetilde{Q}$ the operators $(I + \widetilde{S}_n)/2$, $(I - \widetilde{S}_n)/2$, respectively.

Consider the operator

$$\widetilde{A} = B_n(\widetilde{F}\widetilde{P} + \widetilde{G}\widetilde{Q})B_n + C_n(\widetilde{G}\widetilde{P} + \widetilde{F}\widetilde{Q})C_n,$$

where $B_n = \|\delta_{jk}B\|_{j,k=1}^n$, $C_n = I - B_n$, and B is the operator of multiplication in $L_2(\tilde{\Gamma})$ by the characteristic function of the set Γ. The operator $\widetilde{A}$ is a sum of products of singular integral operators in $L_2^n(\Gamma)$ with piecewise-continuous matrix coefficients. In [1] the symbol $\widetilde{\mathcal{A}}(t,\mu)$ of such an operator is defined. From this definition, in particular, it follows that $\det\widetilde{\mathcal{A}}(t,\mu) = \det\mathcal{A}(t,\mu)$, if $t \in \Gamma$, and $\det\widetilde{\mathcal{A}}(t,\mu) = \det\widetilde{G}(t)\widetilde{F}(t)$, if $t \in \tilde{\Gamma} \setminus \Gamma$.

Let condition (3) be satisfied; then $\det\widetilde{\mathcal{A}}(t,\mu) \neq 0$ $(t \in \Gamma,\ 0 \leq \mu \leq 1)$, hence [1], the operator $\widetilde{A}$ is a Φ-operator in $L_2^n(\tilde{\Gamma})$, and, consequently, the operator A is a Φ-operator in $L_2^n(\Gamma)$.

Conversely, suppose the operator A is a Φ-operator in $L_2^n(\tilde{\Gamma})$; then the operator $A_1 = B_n\widetilde{A}B_n + C_n$ is a Φ-operator in $L_2^n(\tilde{\Gamma})$, and, consequently, $\det\mathcal{A}_1(t,\mu) \neq 0$ $(t \in \tilde{\Gamma},\ 0 \leq \mu \leq 1)$. It is not difficult to verify that if $\det\mathcal{A}_1(t,\mu) \neq 0$, then the symbol $\mathcal{A}_2(t,\mu)$ of the operator $A_2 = B_n + C_n\widetilde{A}C_n$ is also everywhere nondegenerate, and, consequently, the operator A_2 is a Φ-operator in $L_2(\tilde{\Gamma})$. It follows that the operator $\widetilde{A}$ is a Φ-operator in $L_2(\Gamma)$. Since $\det\widetilde{\mathcal{A}}(t,\mu) = \det\mathcal{A}(t,\mu)$ for points $t \in \Gamma$, we have $\det\mathcal{A}(t,\mu) \neq 0$ $(t \in \Gamma,\ 0 \leq \mu \leq 1)$. The theorem is proved.

3. Introduce on the cylinder $\mathfrak{M} = \{(t,\mu) : t \in \Gamma,\ 0 \leq \mu \leq 1\}$ a topology by defining neighborhoods of each point by one of the five equalities:

$$u(\alpha_k, 0) = \{(\alpha_k,\mu) : 0 \leq \mu < \varepsilon\}; \qquad u(\beta_k, 1) = \{(\beta_k,\mu) : \varepsilon < \mu \leq 1\},$$

$$u(t_0, 0) = \{(t,\mu) : |t - t_0| < \delta,\ t < t_0,\ 0 \leq \mu \leq 1\} \cup \{(t_0,\mu) : 0 \leq \mu < \varepsilon\} \qquad (t_0 \neq \alpha_k),$$

$$u(t_0, 1) = \{(t,\mu) : |t - t_0| < \delta,\ t > t_0,\ 0 \leq \mu \leq 1\} \cup \{(t_0,\mu) : \varepsilon < \mu \leq 1\} \qquad (t_0 \neq \beta_k),$$

$$u(t_0, \mu_0) = \{(t_0,\mu) : \mu - \delta_1 < \mu < \mu_0 + \delta_2\} \qquad (\mu_0 \neq 0;\ 1),$$

where $0 < \delta_1 < \mu_0$, $0 < \delta_2 < 1 - \mu_0$, $0 < \varepsilon < 1$, and $t < t_0$ means that on the oriented contour Γ the point t precedes the point t_0.

By $\mathcal{P}_n$ we denote the algebra of matrices of functions of order $2n$ of the form

$$H(t,\mu) = \left(H_{jk}(t,\mu)\right)_{j,k=1}^2,$$

where $H_{jk}(t, \mu)$ are arbitrary matrix-functions of order n satisfying the following conditions: a) the matrix-functions $H_{11}(t, \mu), H_{12}(t, \mu), H_{21}(t, \mu)$ and $H_{22}(t, 1-\mu)$ are continuous on the cylinder $\mathfrak{M}$ with the topology introduced above; b) the matrices $H_{21}(t, \mu)$ and $H_{12}(t, \mu)$ vanish if μ takes one of the values $0, 1$, and t is an arbitrary point of the contour, and also when t takes one of the values α_k, β_k $(k = 1, \dots, N)$, while μ is any number of the interval $0 \le \mu \le 1$.

The algebra $\mathcal{P}$ becomes a Banach algebra if one introduces in it the norm equal to

$$\|H(t, \mu)\| = \sup_{\substack{t\in\Gamma \\ 0\le\mu\le1}} s_1(H(t, \mu)),$$

where the number $[s_1(H(t, \mu))]^2$ for each point of the cylinder $\mathfrak{M}$ denotes the largest eigenvalue of the matrix $H(t, \mu)(H(t, \mu))^*$.

Theorem 2. *Let A_{jl} $(j = 1, \dots, k;\ l = 1, \dots, m)$ be singular integral operators with matrix coefficients from Λ_n, and let $\mathcal{A}_{jl}(t, \mu)$ be their symbols. Then for the operator*

$$A = \sum_{j=1}^{k} \prod_{l=1}^{m} A_{jl} \tag{4}$$

the equality

$$\inf_{T\in\mathfrak{S}_\infty} \|A + T\| = \|\mathcal{A}(t, \mu)\| \tag{5}$$

holds, where

$$\mathcal{A}(t, \mu) = \sum_{j=1}^{k} \prod_{l=1}^{m} \mathcal{A}_{jl}, \tag{6}$$

and the norm on the right-hand side of (5) is the norm in the algebra $\mathcal{P}_n$.

The proof is analogous to the proof of Theorem 2.2 in (^{1}).

The matrix-function (6) is naturally called the **symbol** of the operator (4). From equality (5) it follows that the symbol of an operator A does not depend on the manner in which the operator A is represented in the form (4).

Let $\mathfrak{A}_n$ be the algebra obtained by closing the set of operators of the form (4) in the algebra $\mathfrak{R}_n$. Equality (5) makes it possible to define the symbol $\mathcal{A}(t, \mu)$ for each operator $A \in \mathfrak{A}_n$ as the limit in the algebra $\mathcal{P}_n$ of a sequence of symbols $\mathcal{A}_r(t, \mu)$ of operators A_r of the form (4), converging uniformly to the operator A.

Theorem 3. *The two-sided ideal $\mathfrak{S}_\infty$ of all completely continuous operators acting in $L_2^n(\Gamma)$ is contained in the algebra $\mathfrak{A}_n$, and the quotient algebra $\mathfrak{A}_n/\mathfrak{S}_\infty$ is isomorphic and isometric to the algebra $\mathcal{P}_n$. Under this isomorphism the residue class containing the operator A $(\in \mathfrak{A}_n)$ is mapped to the symbol $\mathcal{A}(t, \mu)$ of the operator A. In order that the operator A $(\in \mathfrak{A}_n)$ be a Φ-operator in $L_2^n(\Gamma)$, it is necessary and sufficient that the condition $\det \mathcal{A}(t, \mu) \neq 0$ hold for all $t \in \Gamma$ and $0 \le \mu \le 1$.*

The proof of this theorem is carried out according to the same scheme as the proof of Theorems 4.1 and 4.2 in (^{1}).

4. Let us establish a formula for computing the index of Φ-operators from the algebra $\mathfrak{A}_n$. Recall that the index of a Φ-operator A is the number $\operatorname{ind} A$, equal to the difference $\dim\ker A - \dim\operatorname{coker} A$.

In this section we additionally assume that the contour Γ can be completed to a closed contour $\hat{\Gamma}$, bounding a connected set $\hat{M}$ of points of the plane and, consequently, consisting of a finite number of closed simple contours of Lyapunov type. We shall also assume that the point $z = 0$ is an interior point of the set $\hat{M}$.

Theorem 4. *Let the operator $A \in \mathfrak{A}_n$, and let the matrix-function*

$$\mathcal{A}(t,\mu) = \left\| H_{jk}(t,\mu) \right\|_{j,k=1}^{2}$$

be its symbol. If $\det \mathcal{A}(t,\mu) \neq 0$ ((t ,

$0 \leqslant \mu \leqslant 1$), then the function

$$f_A(t,\mu) = \begin{cases} \det H_{22}(\alpha_k,\mu)\, H_{22}^{-1}(\alpha_k,0), & \text{for } t = \alpha_k, \\ \det \mathcal{A}(t,\mu) \det H_{22}^{-1}(t,0) H_{22}^{-1}(t,1), & \text{for } t \in \Gamma \text{ and } t \neq \alpha_k, \beta_k, \\ \det H_{22}(\beta_k,\mu) H_{22}^{-1}(\beta_k,1), & \text{for } t = \beta_k \end{cases}$$

is continuous on $\mathfrak{M}$, and

$$\operatorname{ind} A = -\frac{1}{2\pi} \left[\arg f_A(t,\mu)\right]_{\mathfrak{M}}. \tag{7}$$

Let us explain the meaning of the right-hand side of formula (7). If the operator has the form

$$A = \sum_{j=1}^{k} \prod_{l=1}^{m} (F_{jl} P + G_{jl} Q) \qquad (F_{jl}, G_{jl} \in \Lambda_n), \tag{8}$$

then the set of values of the function $f_A(t,\mu)$ consists of a finite number of closed continuous curves, which are naturally oriented: at the points of continuity of all the matrix-functions $F_{jl}(t)$ and $G_{jl}(t)$, motion along the curve $f_A(t,\mu)$ is determined by the change of t along the contour in the positive direction, while along the additional arcs it is determined by the change of μ from 0 to 1. The number $[\arg f_A(t,\mu)]_{\mathfrak{M}}/2\pi$ is equal to the number of turns of the curve $f_A(t,\mu)$ around the origin.

Let $A \in \mathfrak{A}_n$. Then the function $f_A(t,\mu)$ is the uniform limit of a sequence of functions $f_{A_N}(t,\mu)$, where the A_N are operators of the form (8). For sufficiently large N, the number $[\arg f_{A_N}(t,\mu)]_{\mathfrak{M}}$ does not depend on N, and by definition

$$[\arg f_A(t,\mu)]_{\mathfrak{M}} = \lim_{N\to\infty} [\arg f_{A_N}(t,\mu)]_{\mathfrak{M}}.$$

The proof of Theorem 4 is carried out according to the following scheme. Since both sides of formula (7) are continuous functions of the operator A, it is sufficient to establish this formula for operators of the form (9). Let A be an operator of the form (9), and let $L = \Xi(F_{jl}P + G_{jl}Q)$ be its linear extension (see (2)). The operator L is a singular integral operator with matrix coefficients in the space $L_2^r(\Gamma)$, where $r = (mk - k + 1)n$. Let $\widetilde{F}_{jl}(t)$ and $\widetilde{G}_{jl}(t)$ $(t \in \tilde{\Gamma})$ be, respectively, certain matrix-functions coinciding with $F_{jl}(t)$ and $G_{jl}(t)$ on the contour Γ and continuous on the closed set $\tilde{\Gamma} \setminus \Gamma$; then $\operatorname{ind} A = \operatorname{ind}(B_r \widetilde{L} B_r + C_r)$, where $\widetilde{L} = \Xi(\widetilde{F}_{jl}P + \widetilde{G}_{jl}Q)$. Thus, the problem of computing the index of the operator A is reduced to the problem of computing the index of the operator $\widetilde{A} = B_r \widetilde{L} B_r + C_r$, acting in $L_2^r(\tilde{\Gamma})$, which was solved in (1). From the formula for the index of the operator $\widetilde{A}$ given in (1), formula (8) is easily derived.

5. Let $t_1, \dots, t_r$ $(r = 0, 1, \dots)$ be fixed points on the contour Γ, distinct from the endpoints of the open arcs. By $\mathfrak{A}_n(t_1, \dots, t_r)$ we denote the algebra obtained by closing in the algebra $\mathfrak{A}_n$ the set of operators of the form (8) for which the matrix-functions $F_{jl}(t)$ and $G_{jl}(t)$ are continuous at all points of the contour Γ, except possibly the points $t_1, \dots, t_r$. In this case the factor algebra $\mathfrak{A}(t_1, \dots, t_r)/\mathfrak{S}_\infty$ is isomorphic and isometric to the algebra of matrix-functions defined on the contour obtained from the contour Γ by splitting each point t_k $(k = 1, \dots, r)$ into t_k^- and t_k^+ and adding intervals with endpoints t_k^- and t_k^+, and also by adding intervals to each of the endpoints α_k and β_k of the open arcs.

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