

# ON PERIODIC GROUPS WITH CERTAIN FINITENESS CONDITIONS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **ON PERIODIC GROUPS WITH CERTAIN FINITENESS CONDITIONS**

*(Presented by Academician V. M. Glushkov, May 4, 1970)*

In accordance with the work <sup>(1)</sup>, a locally finite group with extremal Sylow  $p$ -subgroups (for a given prime number  $p$ ) is called an  $S_{pF}$ -group. In particular, if a locally finite group is an  $S_{pF}$ -group for every prime divisor  $p$  of the orders of its elements, then it is called an  $SF$ -group.

In the present paper a result is formulated which gives a necessary and sufficient condition for the invariance of a maximal complete abelian 2-subgroup in an  $S_2F$ -group with an infinite Sylow 2-subgroup (see Theorem 1). Using Theorem 1, it is easy to prove that an arbitrary  $SF$ -group is an extension of an abelian group by means of an  $SF$ -group with finite Sylow 2-subgroups (see Theorem 2). Hence, and from Theorems 3, 4 and a result of S. N. Chernikov <sup>(2)</sup>, it follows that questions 2.1, 8.1, 8.2 posed in the survey <sup>(3)</sup>, as well as question 23 from <sup>(4)</sup>, need only be considered for  $SF$ -groups with finite Sylow 2-subgroups; and since locally soluble  $SF$ -groups have been studied up to  $SF$ -groups with finite Sylow  $p$ -subgroups <sup>(5)</sup>, and locally finite groups of odd order are locally soluble <sup>(6)</sup>, the significance of Theorems 2, 3, 4 in connection with the consideration of the indicated questions becomes clear.

With the aid of Theorems 2, 3 the author proves that, for locally finite groups, the minimality condition for abelian subgroups implies the minimality condition for subgroups, and that Sylow  $p$ -subgroups in an  $SF$ -group are conjugate. Consequently, questions 2.1, 8.2 from <sup>(3)</sup> are equivalent.

The conjugacy of Sylow  $p$ -subgroups in locally finite groups with the minimality condition for subgroups was first established in the work <sup>(7)</sup>.

In Section 2 a class of  $p$ -groups is singled out, called by the author biprimatively finite; for  $p = 2$  these coincide with 2-groups, and to this class of groups Blackburn's result on Chernikov  $p$ -groups <sup>(8)</sup> is generalized (see Theorem 8 of the present work).

An example of a group constructed by P. S. Novikov and S. I. Adian <sup>(9)</sup> shows that Theorem 8 cannot be generalized to arbitrary  $p$ -groups, even under the condition of finiteness of all abelian subgroups in the group.

At the same time, it follows from a result of E. S. Golod <sup>(10)</sup> that a biprimatively finite  $p$ -group need not be locally finite.

As a corollary of Theorem 8 we obtain that a biprimatively finite  $p$ -group (and for  $p = 2$ , an arbitrary 2-group) satisfying the minimality condition for abelian subgroups is a Chernikov  $p$ -group (see Corollary 1).

Corollary 1 for  $p = 2$  was stated without proof by S. P. Strunkov in the abstracts of the Batumi Symposium, 1967, and for  $p$ -groups in which any two elements generate a finite group it was proved by V. G. Viljaper <sup>(11)</sup>.

1. A group containing (not containing) involutions will be called a group of even (odd) order.

$C_G(M)$  is the centralizer of the set  $M$  of elements of the group  $G$ .

$N_G(A)$  is the normalizer of the subgroup  $A$  in the group  $G$ .

$\pi(G)$  is the set of prime divisors of the orders of elements of the group  $G$ .

If  $A$  and  $B$  are subsets of the group  $G$ , then  $AB$  is the set of elements of  $G$  of the form  $g = ab$  ( $a \in A$ ,  $b \in B$ ).

$A\lambda B$  is the semidirect product of the groups  $A$  and  $B$ , where  $A$  is a normal divisor in  $A\lambda B$ .

An element  $g$  of the group  $G$  is called strictly real with respect to the involution  $i$  from  $G$  if  $igi = g^{-1}$ .

By the rank of a group we shall mean the special rank in the sense of Mal'cev <sup>(12)</sup>.

**Definition 1.** A Sylow  $p$ -subgroup  $P$  of a countable locally finite group  $G$  is called **projective** in  $G$  if the group  $G$  has a chain of finite subgroups

$$N_1 \subset N_2 \subset \dots \subset N_s \subset \dots,$$

whose union coincides with  $G$ , such that  $P \cap N_s$  ( $s = 1, 2, \dots$ ) is a Sylow  $p$ -subgroup in  $N_s$ .

Every countable locally finite group  $G$ , for any  $p \in \pi(G)$ , has a projective Sylow  $p$ -subgroup. This is not hard to prove by using the well-known method of projections <sup>(13)</sup>.

**Definition 2.** We shall say that an  $S_{pF}$ -group has a  $p$ -complete part if a maximal complete abelian  $p$ -subgroup in it is invariant. In particular, if the Sylow  $p$ -subgroups of an  $S_{pF}$ -group are finite, then we shall agree to regard the identity subgroup as the  $p$ -complete part.

If  $G$  is an  $S_{pF}$ -group, then an element  $g$  of order  $p$  from  $G$  will be called an element of the first kind if  $C_G(g)$  has a  $p$ -complete part, and an element of the second kind otherwise.

**Definition 3.** The rank of the  $p$ -complete part of some projective Sylow  $p$ -subgroup of a countable  $S_{pF}$ -group  $G$  will be denoted by  $t_p(G)$  and will be called the  $p$ -complete rank of the group  $G$ . The index of the  $p$ -complete part in a subgroup  $P$  of the group  $G$  will be called the  $p$ -complete index of the group  $G$  and will be denoted by  $l_p(G)$ .

If the Sylow  $p$ -subgroups of the  $S_{pF}$ -group  $G$  are finite (in particular, if  $G$  contains no elements of order  $p$ ), then we shall agree to set  $t_p(G) = 0$ .

It is easy to prove that the  $p$ -complete rank of a countable  $S_{pF}$ -group  $G$  does not depend on the choice of the projective Sylow  $p$ -subgroup, and if for some subgroup  $H$  of  $G$   $t_p(H) = t_p(G)$ , then  $l_p(H) \leq l_p(G)$ , and also the  $p$ -complete rank of a subgroup does not exceed the  $p$ -complete rank of the group.

**Theorem 1.** *Let  $G$  be an  $S_2F$ -group with an infinite Sylow 2-subgroup, satisfying the following condition:*

) if  $H$  is a subgroup of odd order and  $N_G(H)$  contains a quasicyclic 2-subgroup  $Q$ , then  $Q \subset C_G(H)$ . Then a maximal complete abelian 2-subgroup of  $G$  is invariant in  $G$ .\*

Theorem 1 is proved by contradiction, i.e., it is assumed that there exist  $S_2F$ -groups (without impairing the generality of the argument, they may be assumed countable) satisfying condition \*), but not having 2-complete parts.

In the class of all such countable groups one chooses the subset of groups of least 2-complete rank, and in this subset the class of all groups of least 2-complete index; this class is denoted by  $K_2$ .

In the first part of the proof of Theorem 1 it is established that in  $K_2$  there exists a group in which every involution is an element of the first kind. Here the main tool of the proof is Glauberman's theorem <sup>(14)</sup> (and if one does not use this theorem, then Grün's theorem <sup>(15)</sup>).

In the second part of the proof of Theorem 1, instead of  $K_2$  one considers the subset  $K'_2$  of all groups from  $K_2$  in which every involution is an element of the first kind.

It is proved that  $K'_2$  has a group such that it is either isomorphic to a group of type  $PSL(2, T)$  over a field  $T$  of odd characteristic, or has-

gives a subgroup, some infinite factor group of which is isomorphic to one of the groups:  $SL(2, Q)$ , the Suzuki group  $S(Q)$ , or the unitary group  $U_3(Q)$ , where  $Q$  is a field of even characteristic.

In the first case we obtain a contradiction with condition \*, since in a group of type  $PSL(2, T)$  with an infinite Sylow 2-subgroup and over a field of odd characteristic there exists an abelian subgroup  $B$  of odd order, and in its normalizer there is a quasicyclic 2-subgroup which does not centralize the subgroup  $B$ .

In the second case we obtain a contradiction with the extremality of the Sylow 2-subgroup in an  $S_2F$ -group, since in infinite groups of type  $SL(2, Q)$ ,  $S(Q)$ ,

$U_3(Q)$  over a field of even characteristic, the Sylow 2-subgroups are not extremal.

The main tools in the second part of the proof of Theorem 1 are the result of Gorenstein and Walter <sup>(16)</sup>, the result of Suzuki <sup>(17)</sup> on finite groups with an independent Sylow 2-subgroup, Theorem 14.3.1 of <sup>(15)</sup>, and the Brauer–Suzuki result <sup>(18)</sup>.

**Theorem 2.** *An SF-group is an extension of an abelian group by an SF-group with finite Sylow 2-subgroups.*

**Theorem 3.** *A locally finite group with the minimality condition for (abelian) subgroups is an extension of an abelian group by a group with finite Sylow 2-subgroups and the minimality condition for (abelian) subgroups.*

**Theorem 4.** *If an infinite simple locally finite group of finite rank exists, then its Sylow 2-subgroups are finite.*

**Theorem 5.** *For locally finite groups, the minimality condition for abelian subgroups implies the minimality condition for subgroups.*

The question of the validity of Theorem 5 for arbitrary periodic groups remains open.

In the proof of Theorem 5, as well as of Theorem 7, together with Theorem 3 the following lemma is used.

**Lemma.** *Let  $G$  be a finite group,  $H$  a proper subgroup of even order coinciding with its normalizer,  $j \in G$  an involution and  $j \notin H$ ,  $D(j)$  the subgroup of  $H$  generated by all elements which are strictly real with respect to  $j$ , and  $i$  some involution from  $H$ .*

*If the subgroup  $H$  is 2-transversal to its conjugates, i.e. intersects any other subgroup conjugate to it in a subgroup of odd order, then*

- 1) *the subgroup  $D(j)$  is of odd order, and for every strictly real element  $a \neq 1$  of  $H$  with respect to  $j$ , the centralizer  $C_G(a)$  is of odd order;*
- 2) *in each adjacent class  $hC_H(i)$  ( $h \in H$ ) there exists one and only one element from  $D(j)$  which is strictly real with respect to  $j$ , and  $H = D(j)C_H(i)$ ;*
- 3) *no Sylow  $p$ -subgroup of  $D(j)$  is centralized by the involution  $j$ ;*
- 4) *if  $D = H \cap jHj$ , then  $D\lambda\langle j \rangle$ .*

**Theorem 6.** *If in a locally finite group the Sylow 2-subgroups are extremal and the subgroups of odd order are finite, then the group itself is extremal.*

**Theorem 7.** *The Sylow  $p$ -subgroups of an SF-group are conjugate.*

2. Let  $G$  be a periodic group, and  $p$  some prime number, satisfying the following condition:

If  $H$  is a subgroup of  $G$ ,  $N$  its invariant extremal subgroup, then in the factor group  $H/N$  any two elements of order  $p$  (if  $H/N$  has such elements) generate a finite group.

In this case we shall call the group  $G$  **biprimatively finite relative to  $p$** . If  $G$  is biprimatively finite relative to every  $p \in \pi(G)$ , then we shall call the group  $G$  **biprimatively finite**.

Every periodic group of odd order is biprimatively finite with respect to 2.

On the other hand, as the example of E. S. Golod<sup>(10)</sup> shows, there exist biprimatively finite  $p$ -groups (for any  $p$ ) that are not locally finite groups.

**Theorem 8.** If a biprimatively finite  $p$ -group (and for  $p = 2$  an arbitrary 2-group) possesses a finite maximal elementary abelian subgroup, then the group is extremal.

**Corollary 1.** If a biprimatively finite  $p$ -group (and for  $p = 2$  an arbitrary 2-group) satisfies the minimality condition for abelian subgroups, then the group is extremal.

**Corollary 2.** A biprimatively finite  $p$ -group (and for  $p = 2$  an arbitrary 2-group) is finite if and only if it is generated by a finite number of generators and some maximal elementary abelian subgroup of it is finite.

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## CITED LITERATURE

- <sup>1</sup> V. P. Shunkov, *Sibirsk. matem. zhurn.*, **8**, No. 1, 213 (1967).
- <sup>2</sup> S. N. Chernikov, *Matem. sborn.*, **7**, 49, 539 (1940).
- <sup>3</sup> S. N. Chernikov, *UMN*, **14**, issue 5, 45 (1959).
- <sup>4</sup> *Kourovka Notebook*, Novosibirsk, 1965.
- <sup>5</sup> M. I. Kartapolov, *Sibirsk. matem. zhurn.*, **2**, No. 6, 853 (1961).
- <sup>6</sup> W. Feit, I. G. Thompson, *Pacific J. Math.*, **13**, No. 3, 775 (1963).
- <sup>7</sup> R. Baer, *Math. Ann.*, **150**, No. 1, 1 (1963).
- <sup>8</sup> N. Blackburn, *J. Math.*, **6**, 421 (1962).
- <sup>9</sup> P. S. Novikov, S. I. Adyan, *Izv. AN SSSR, ser. matem.*, **32**, No. 1, 2, 3 (1968).
- <sup>10</sup> E. S. Golod, *Izv. AN SSSR, ser. matem.*, **28**, 273 (1964).
- <sup>11</sup> B. T. Vilyatser, *UMN*, **13**, No. 2, 163 (1958).
- <sup>12</sup> A. I. Mal' tsev, *Matem. sborn.*, **22**, 273 (1948).
- <sup>13</sup> A. G. Kurosh, *Group Theory*, 3rd ed., "Nauka," 1967.
- <sup>14</sup> G. Glauberman, *J. Algebra*, **4**, 403 (1966).
- <sup>15</sup> M. Hall, *Group Theory*, IL, 1962.
- <sup>16</sup> D. Gorenstein, J. Walter, *J. Math.*, **6**, 553 (1962).

<sup>17</sup> M. Suzuki, *Ann. Math.*, **80**, 58 (1964).

<sup>18</sup> R. Brauer, M. Suzuki, *Proc. Nat. Acad. Sci. U.S.A.*, **45**, 1757 (1959).

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