ON THE TOPOLOGY OF UNIVERSE MODELS

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Abstract

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ON THE TOPOLOGY OF UNIVERSE MODELS

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1°. An important question in cosmology is the problem of singling out those classes of manifolds that can serve as models of the Universe. For a correct consideration of Einstein' s equations it is necessary first, proceeding from physical considerations, to choose the manifold on which they will be defined. On the basis of the most general physical laws, a number of principles have been proposed that restrict the topology of Universe models independently of their concrete form*.

($\mathfrak{B}$) $\mathfrak{M}^4$ is a smooth manifold on which one can define an indefinite metric of signature $(1, 3)$.

($\mathfrak{G}_1$) All timelike and isotropic vectors can be divided into two classes $\{+\}$ and $\{-\}$ in such a way that, if $\xi \in \{+\}$, then $-\xi \in \{-\}$, and conversely, and if $\xi_n \in \{+\}\,(\{-\})$, then $\lim \xi_n \in \{+\}\,(\{-\})$.

($\mathfrak{G}_2$) $\mathfrak{M}^4$ contains no closed timelike lines.

($\mathfrak{G}_3$) On $\mathfrak{M}^4$ there exists a smooth point function $f(P) = t$ such that in some local coordinates t coincides with the time coordinate.

(ZN) $\mathfrak{M}^4$ contains no nonorientable spaces $\mathfrak{M}^3$.

2°. The class $\mathfrak{B}$ is narrower than the class of all smooth manifolds, since an indefinite metric can be defined not on all manifolds (a necessary condition for this is equality to 0 of the Euler characteristic ([3])). The principle $\mathfrak{B}$ is the most fundamental of all these principles. The principles $\mathfrak{G}_1$–$\mathfrak{G}_3$ were proposed by K. Gödel ([7]) as different formulations of the principle of causality. The principle ZN was proposed by Ya. B. Zel' dovich and I. D. Novikov as a restriction following from CP-noninvariance ([1,2]).

3°. Repeating literally the reasoning of ([1,2]), one can obtain a restriction on the topology of Universe models following from CT-noninvariance. If CP-noninvariance makes it possible to distinguish the two possible orientations of $\mathfrak{M}^3$, then CT-noninvariance distinguishes the two possible directions of time at points of $\mathfrak{M}^3$, i.e., the two possible orientations of the timelike vectors defined on $\mathfrak{M}^3$. That is, upon transport along a closed contour on $\mathfrak{M}^3$, a timelike vector must not change its orientation, which coincides with the definition of two-sidedness of $\mathfrak{M}^3$ ([3]). Thus, the following restriction on the topology of Universe models is proposed:

($\mathfrak{MB}$) $\mathfrak{M}^4$ cannot contain one-sidedly embedded $\mathfrak{M}^{3}$' s.

Let us note that, if ZN is an internal restriction with respect to $\mathfrak{M}^3$, then $\mathfrak{MB}$ is an external one.

4°. Let us give an illustration of the use of the principles ZN and $\mathfrak{MB}$: we construct the manifold $\mathfrak{M}^4$—the direct product of the Klein bottle with the plane $(p; r)$. Recall that the direct product of two manifolds with coordinates $(u; v)$ and $(p; r)$ is a manifold whose points have coordinates $(u; v; p; r)$. The Klein bottle is obtained by gluing, in a square in the plane $(u; v)$, two sides with preservation of orientation and two with reversal of it, as a result of which a coordinat—

* The following notation is used below: $\mathfrak{M}^4$ is space-time; $\mathfrak{M}^3$ is space (concerning this concept see §8); R^1 is the real line. The letters numbering the principles also denote the classes of manifolds singled out by them.

ian grid with "periodic" coordinates:

$$(u; v + 2\pi) = (u; v); \qquad (u + 2\pi; 2\pi - v) = (u; v). \tag{1}$$

Let us specify on this $\mathfrak{M}^4$ two metrics:

$$ds^2 = dp^2 - du^2 - dv^2 - dr^2, \tag{2}$$

$$ds^2 = du^2 - dv^2 - dp^2 - dr^2. \tag{3}$$

In (2), $\mathfrak{M}^3$ is the direct product of a Klein bottle with R^1—nonorientable—and (2) does not satisfy ZN; in (3), $\mathfrak{M}^3$ is the product of a cylinder and a line—one-sidedly embedded in $\mathfrak{M}^4$ (for the proof see ([3])), and (3) does not satisfy $\mathfrak{MB}$. The idea of the examples is borrowed from ([6,8]). The fact that (3) does not satisfy $\mathcal{G}_2$ is immaterial ([9]).

5°. Let us note that ZN and $\mathfrak{MB}$ are physically proved by direct experiments.

6°. What is the relation between the principles considered above? We shall show that, by strengthening $\mathcal{G}_3$, one can obtain a principle from which $\mathfrak{B}, \mathcal{G}_1, \mathcal{G}_2$, and $\mathfrak{MB}$ follow.

($\mathfrak{KB}$) On $\mathfrak{M}^4$ one can define a smooth function of a point $f(P) = t$ such that, in some local coordinates, t coincides with the time coordinate, t ranges over the interval (a, b), and $f^{-1}(a) = \mathfrak{N}_1^{3*}$ and $f^{-1}(b) = \mathfrak{N}_2^3$ are three-dimensional closed smooth manifolds.

Then $f(P)$ is a Morse function of the manifold $\mathfrak{M}^4$ without critical points and $\mathfrak{N}_1^3$ ([4]) is diffeomorphic to $\mathfrak{N}_2^3$, and

$$\mathfrak{M}^4 = \mathfrak{N}^3 \times R^1. \tag{4}$$

As is easy to verify, the principles $\mathfrak{B}, \mathcal{G}_1, \mathcal{G}_2, \mathfrak{MB}$ are in this case trivially fulfilled, while ZN forbids nonorientability of $\mathfrak{N}^3$. That is, $\mathfrak{KB}$ reduces the problem of the topology of models of the Universe to the three-dimensional one, where it is already posed as the problem of different "gluings" of space, i.e., as the problem of Heegaard diagrams ([3]). Let us note that all cosmological models known to the author that have physical meaning, as well as solutions of the equations of the general theory of relativity, satisfy $\mathfrak{KB}$. It may be supposed that $\mathfrak{KB}$ follows from $\mathcal{G}_1 - \mathcal{G}_3, \mathfrak{B}$, and $\mathfrak{MB}$. It is also interesting to find out precisely which manifolds enter into the classes considered above.

7°. The principles formulated above admit a natural generalization to the case of arbitrary dimension and signature; here classes $\mathfrak{B}_p^i$ arise (i.e., the class of n-dimensional smooth manifolds on which one can specify a metric of signature $(p; q)$), etc. However, their intersection is nonempty only for some n, p, q, so that, perhaps, n, p, and q can be obtained from these principles. Let us note that if a violation of CPT were observed, it would not impose restrictions analogous to ZN and $\mathfrak{MB}$, since PT does not change the orientation of $\mathfrak{M}^4$.

8°. In conclusion we give various definitions of $\mathfrak{M}^3$:

($\mathfrak{C}$) $\mathfrak{M}^3$ is any spacelike three-dimensional submanifold embedded in $\mathfrak{M}^4$ ([7]).

($\mathfrak{GM}$) $\mathfrak{M}^3$ is a spacelike three-dimensional submanifold that every timelike geodesic intersects once and only once ([5,7]).

In some works of Ya. B. Zel' dovich and A. L. Zel' manov, for $\mathfrak{M}^4$ representable in the form (4), the following definition of space is meant:

(ZZ) $\mathfrak{M}^3$ is a spacelike submanifold diffeomorphic to $\mathfrak{N}^3$.

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* $f^{-1}(a) = \mathfrak{N}_1^3$ means that for all τ sufficiently close to a, $f^{-1}(\tau)$ is diffeomorphic to $\mathfrak{N}_1^3$.

For the definition G it has not been proved that at least one M^4 satisfies ZN; the difficulties of GM are discussed in ([5]). In the case of justification of KP, the definition ZZ appears preferable to the author.

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Note: Figure translations are in progress. See original paper for figures.