ON DEEWEL HOMOLOGIES OF METRIC COMPACTA

1970

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Abstract

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UDC 513.836

MATHEMATICS

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ON DEEWEL HOMOLOGIES OF METRIC COMPACTA

(Presented by Academician P. S. Aleksandrov on 27 IV 1970)

Let X be a topological space.

Definition. A precosheaf of Abelian groups (or modules) with base X is a covariant functor from the category of open sets of the space X to the category of Abelian groups (or modules).

Let A be a precosheaf with base X. Denote by $\check{H}_*(X; A)$ the Aleksandrov-Čech homologies, obtained as the inverse limit of the homologies of open covers of the space X with coefficients in A ([5]). As is known, an essential drawback of Aleksandrov-Čech homology, which makes its application difficult, is the absence of exactness in the homology sequence of a pair or in the homology sequence corresponding to an exact triple of coefficients.

A fundamentally different approach to the definition of homology, taking into account derived functors of the functor of the projective (inverse) limit, was given in Deewel' s work ([4]). Deewel' s basic constructions, however, are given not in the category of topological spaces but in the category of ordered sets. This leads to difficulties in clarifying their nature in concrete categories of topological spaces and their relation to other theories (for example, to ordinary homology in the category of noncompact polyhedra). However, in one case, namely in the category of metric compacta, the author shows that for constant coefficients his homologies are isomorphic to Steenrod homologies* ([4]). Deewel homologies satisfy the requirement of exactness; to every exact triple of precosheaves over X

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

there corresponds an exact homology sequence

$$\cdots \to H_n(X; A_1) \to H_n(X; A_2) \to H_n(X; A_3) \to H_{n-1}(X; A_1) \to \cdots .$$

Deewel homologies may be defined in the following way. Let $I^*(C_*)$ be an injective resolution of some spectrum C_* of chain complexes of open covers of the space X with coefficients in the precosheaf A. Then $H_n(X;A) = H_n(\lim_{\leftarrow} I^*(C_*))$, where $\lim_{\leftarrow} I^*(C_*)$ is regarded as a double complex. This approach makes it possible to obtain certain further properties of Deewel homologies of metric compacta with coefficients in precosheaves.

From consideration of the spectral sequence of the double complex $\lim_{\leftarrow} I^*(C_*)$ there follows the following theorem, giving a relation between Deewel homologies and Aleksandrov-Čech homologies.

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* Up to indices, since Steenrod homologies, defined as n-dimensional, turn out to be $(n-1)$-dimensional (and coincide with the ordinary ones in the category of compact polyhedra). Later Borel and Moore ([3]) constructed an exact homology theory in the category of locally compact spaces. As shown in ([2]) (where a description is given of the exact homology theory with constant coefficients going back to Steenrod), the Borel-Moore homologies are isomorphic to Steenrod homologies (allowing for the error in the index) in the category of compacta under the additional restriction that the coefficient module have a finite number of generators (without this restriction it is not quite correct, see ([2]), pp. 94 and 109).

Theorem 1. Let X be a metric compactum, and let A be a precosheaf with base X. Then, for all n, there exist exact sequences

$$0 \to \lim_{\{\mathfrak{M}\}}^{(1)} H_{n+1}(\mathfrak{M};A) \to H_n(X;A) \xrightarrow{\varphi} \widetilde{H}_n(X;A) \to 0,$$

natural in A, where $\lim^{(1)}_{\{\mathfrak{M}\}}(\cdot)$ denotes the first derived functor of the inverse-limit functor $\lim_{\{\mathfrak{M}\}}(\cdot)$, considered over the directed set $\{\mathfrak{M}\}$ of all open covers of the space X (for derived functors of the inverse-limit functor, see ([6])).

In the case of constant coefficients this theorem was formulated by Roos ([6]).

Corollary 1. If A is a precosheaf of compact Abelian groups or finite-dimensional vector spaces, then

$$H_*(X;A) = \widetilde{H}_*(X;A).$$

Corollary 2. For any precosheaf of Abelian groups (or modules) A, the homology groups $H_n(X;A) = 0$ for $n < -1$. If A is a constant precosheaf, then $H_{-1}(X;A) = 0$.

Corollary 3. If X is a metric compactum of dimension k, then

$$H_n(X;A) = 0 \quad \text{for } n > k$$

and

$$H_k(X;A) = \widetilde{H}_k(X;A).$$

From consideration of the second spectral sequence of the double complex $\lim_{\leftarrow} I^*(C_*)$ the following result follows.

Theorem 2. Let X be a metric compactum, let $\{\mathfrak{M}_n, p_n^{n+1}\}_{n\in Z^+}$ be a refining spectrum of open covers of the metric compactum (Z^+ are the natural numbers), and let A be a precosheaf with base X. Then there exists an exact sequence

$$\cdots \to H_k\Big(\lim_{\leftarrow n} C_*(\mathfrak{M}_n;A)\Big) \to H_k(X;A) \to H_{k+1}\Big(\lim_{\leftarrow n} C_*(\mathfrak{M}_n;A)\Big) \to$$

$$\to H_{k-1}\Big(\lim_{\leftarrow n} C_*(\mathfrak{M}_n;A)\Big) \to \cdots$$

natural in A, where $C_*(\mathfrak{M}_n;A)$ is the chain complex of the cover $\mathfrak{M}_n$ with coefficients in A.

In the case of constant coefficients $\lim^{(1)}_{\leftarrow n} C_*(\mathfrak{M}_n;A) = 0$, and therefore Theorem 2 gives an isomorphism of Deheuvels homology with the homology considered in ([2]).

The next theorem says that, in the case when X is a metric compactum without isolated points, Deheuvels homology is a universal extension of Alexander–Čech homology among homologies satisfying the exactness axiom and certain other natural requirements.

Theorem 3. Let $\overline{H}_*(X;\cdot)$ be any homology theory with coefficients in precosheaves of Abelian groups, defined on a metric compactum X without isolated points and satisfying the exactness axiom. Suppose that

$$H_{-1}(X;A) = 0$$

for any precosheaf of compact Abelian groups A. Then, if there is a natural transformation of homologies

$$f : \overline{H}_*(X;\cdot) \to \widetilde{H}_*(X;\cdot),$$

there exists a unique natural transformation

$$\bar{f} : \overline{H}_*(X;\cdot) \to H_*(X;\cdot),$$

for which the diagram is commutative:

$$\begin{array}{ccc} H_*(X;\cdot) & \xrightarrow{\varphi} & \widetilde{H}_*(X;\cdot) \\ & \nwarrow_{\bar{f}} \;\; \nearrow^{f} & \\ & \overline{H}_*(X;\cdot) & \end{array}$$

Theorem 3 is also true for any finite-dimensional compacta (possibly with isolated points).

It is well known that the cohomology of Alexander-Čech with coefficients in presheaves of Abelian groups (or modules) are the right pro-

derived functors of the functor $H^0(X;\,\cdot)$. An analogous theorem holds for Deheuvels homology.

Theorem 4. *Let X be a metric compactum without isolated points. Then the Deheuvels homology $H_n(X;\,\cdot)$, $n=-1,0,1,2,...$, with coefficients in precosheaves of abelian groups (or modules) is isomorphic to the $(n+1)$-st derived functors of the functor $H_{-1}(X;\,\cdot)$.*

The following theorem shows that the assertions of Corollaries 1, 3 and the requirement of exactness completely determine Deheuvels homology on finite-dimensional compacta.

Theorem 5. *Let X be a finite-dimensional metric compactum. Let $\bar{H}_*(X;\,\cdot)$ be an exact homology theory with coefficients in precosheaves of abelian groups, coinciding with the Alexander-Čech homology theory on the full subcategory whose objects are precosheaves of compact abelian groups. Suppose, moreover, that $\bar{H}_n(X;\,\cdot)=0$ for all sufficiently large n. Then the theory $\bar{H}_*(X;\,\cdot)$ is isomorphic to the homology theory $H_*(X;\,\cdot)$.*

For Deheuvels homology there are results that are analogues of the theorems on the spectral sequence of a continuous mapping and the spectral sequence of a covering in the case of cohomology with coefficients in sheaves ([1]).

Let $\pi : X \to Y$ be a continuous mapping of metric compacta; A a precosheaf with base X. Denote by $\mathcal{H}_q(F;A)$ the precosheaf with base Y which assigns to each open set v the group $H_q(X;A^{\pi^{-1}(v)})$, where $A^{\pi^{-1}(v)}$ is the precosheaf with base X defined by the relation

$$A^{\pi^{-1}(v)}(U)=A(U\cap\pi^{-1}(v)).$$

Theorem 6. *There exists a spectral sequence for which*

$$E^2_{p,q}=H_q(Y;\mathcal{H}_q(F;A)),$$

and the term E^∞ is associated with the homologies $H_*(X;A)$ of a suitable filtration.

Let $\mathfrak{M} = \{M_i\}_{i \in I}$ be a finite open covering of the metric compactum X, and let A be a precosheaf with base X. For every finite subset $S \subset I$ denote by M_S the set $\bigcap_{i \in S} M_i$. Consider on the nerve of the covering $\mathfrak{M}$ the coefficient system $\mathcal{H}_q(A) : S \to H_q(X; A^{M_S})$, where A^{M_S} is the precosheaf with base X defined by the relation $A^{M_S}(U) = A(U \cap M_S)$.

Theorem 7. *There exists a spectral sequence for which*

$$E^2_{p,q} = H_p(\mathfrak{M}; \mathcal{H}_q(A)),$$

and the term E^∞ is associated with the homologies $H_*(X; A)$.

Corollary 4. *Let $\mathfrak{M} = \{M_i\}_{i \in I}$ be a finite open covering of the metric compactum X, and let A be a precosheaf with base X. If $H_q(X; A^{M_S}) = 0$ for $q \neq 0$ and for every simplex S of the nerve of the covering, then*

$$H_n(\mathfrak{M}; \mathcal{H}_0(A)) = H_n(X; A).$$

I express my deep gratitude to E. G. Sklyarenko for his assistance.

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Received
22 IV 1970

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Note: Figure translations are in progress. See original paper for figures.