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ON THE CONVERGENCE OF A SEQUENCE OF LINEAR CONTINUOUS MAPPINGS OF ANALYTIC SPACES

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In the present note, necessary and sufficient conditions are found for the convergence of sequences of linear continuous mappings of analytic spaces ([1–6]).

The starting point is the following

Theorem 1. Let X be a barrelled space ([7]), Y a complete locally convex space, and let $\{A_n\}$ be a sequence of linear continuous mappings of X into Y. In order that $\{A_n f\}$ converge in Y for every $f \in X$, it is necessary and sufficient that the family $\{A_n\}$ be equicontinuous ([7]) and that $\{A_n g\}$ converge for all g from a total set $\mathfrak{G} \subset X$.

Let $\mathfrak{A}$ ($\mathfrak{A}'$) be the projective limit ([8,9]) of a sequence of Banach spaces $\mathfrak{B}_n$ ($\mathfrak{B}'_n$) with increasing norms $\| \, \|_n$ ($\| \, \|'_n$), and let $\mathfrak{A}$ ($\mathfrak{A}'$) be everywhere dense in each $\mathfrak{B}_n$ ($\mathfrak{B}'_n$). Let $\overline{\mathfrak{A}}$ ($\overline{\mathfrak{A}}'$) be the inductive limit ([8,9]) of an increasing sequence of Banach spaces $\mathfrak{B}_n$ ($\mathfrak{B}'_n$) with decreasing norms $\| \, \|_n$ ($\| \, \|'_n$).

Theorem 2. In order that the sequence $\{A_n\}$ of linear continuous mappings of $\mathfrak{A}$ into $\mathfrak{A}'$ converge in $\mathfrak{A}'$ on every $f \in \mathfrak{A}$, it is necessary and sufficient that: 1) $\sup_k \|A_k\|_{n(m),m} = N_m < \infty$ $(m = 1, 2, ...)$ ($\|A_k\|_{n,m} = \|A_k\|_{\mathfrak{B}n,\mathfrak{B}'_m}$ is the norm of the mapping A_k from $\mathfrak{B}_n$ into $\mathfrak{B}'_m$); 2) $\{A_k g\}$ converge for all g from a total set $\mathfrak{G} \subset \mathfrak{A}$.

Theorem 3. In order that the sequence of linear continuous mappings $\{A_n\}$ of the space $\overline{\mathfrak{A}}$ into $\mathfrak{A}'$ (under the assumption that $\mathfrak{B}'_m$ is completely continuously embedded in $\mathfrak{B}'_{m+1}$ for all m) converge in $\mathfrak{A}'$ on every $f \in \overline{\mathfrak{A}}$, it is necessary and sufficient that: 1) for some n and m, $\sup_k \|A_k\|_{n,m} = N < \infty$; 2) $\{A_k g\}$ converge for all g from a total set $\mathfrak{G} \subset \overline{\mathfrak{A}}$.

Theorem 4. In order that the sequence $\{A_n\}$ of linear continuous mappings of $\overline{\mathfrak{A}}$ into $\overline{\mathfrak{A}}'$ (under the assumption that $\mathfrak{B}'_m$ is completely continuously embedded in $\mathfrak{B}'_{m+1}$ for all m) converge in $\overline{\mathfrak{A}}'$ on every $f \in \overline{\mathfrak{A}}$, it is necessary and sufficient that: 1) $\sup_k \|A_k\|_{n,m(n)} = N_n < \infty$ $(n = 1, 2, ...)$; 2) $\{A_k g\}$ converge for all g from a total set $\mathfrak{G} \subset \overline{\mathfrak{A}}$.

Theorem 5. In order that the sequence $\{A_n\}$ of continuous linear mappings of $\overline{\mathfrak{A}}$ into $\overline{\mathfrak{A}}'$ converge in $\overline{\mathfrak{A}}'$ on every $f \in \overline{\mathfrak{A}}$, it is necessary and sufficient that: 1) $\sup_k \|A_k\|_{n,m} = N_{n,m} < \infty$ $(n, m = 1, 2, ...)$; 2) $\{A_k g\}$ converge for all g from a total set $\mathfrak{G} \supset \overline{\mathfrak{A}}$.

In considering the space $\overline{\mathfrak{A}}'$, we used only the following consequence of the complete continuity of the embedding of $\mathfrak{B}'_m$ in $\mathfrak{B}'_{m+1}$, $m = 1, 2, ...$: a sequence $f_n \to 0$ in $\overline{\mathfrak{A}}'$ if and only if, for some m, $\{f_n\} \subset \mathfrak{B}'_m$ and $\|f_n\|_m \to 0$. This makes it possible to replace one of the analytic spaces by a Banach space.

Theorem 6. In order that the sequence $\{A_n\}$ of linear continuous mappings: 1) $\mathfrak{A}$ into $\mathfrak{B}$; 2) $\mathfrak{B}$ into $\mathfrak{A}'$; 3) $\overline{\mathfrak{A}}$ into $\mathfrak{B}$; 4) $\mathfrak{B}$ into $\overline{\mathfrak{A}}'$ converge

on each element of the corresponding space, it is necessary and sufficient that $\{A_k g\}$ converge, for all g from a total set $\mathfrak{G}$ of the corresponding space, and: 1) for some n

$$\sup_k \|A_k\|_{\mathfrak{B}_n, \mathfrak{B}} = N < \infty;$$

2) for each n

$$\sup_k \|A_k\|_{\mathfrak{B}, \mathfrak{B}_n} = N_n < \infty;$$

3) for each n

$$\sup_k \|A_k\|_{\mathfrak{B}_n, \mathfrak{B}} = N_n < \infty;$$

4) for some n

$$\sup_k \|A_k\|_{\mathfrak{B}, \mathfrak{B}_n} = N < \infty.$$

Denote by $\mathfrak{A}_R$ $(\mathfrak{A}_P)$ the space of analytic functions in the disk $|z| < R$ $(|z| < P)$ and by $\overline{\mathfrak{A}}_R$ $(\overline{\mathfrak{A}}_P)$ the space in the closed disk $|z| \le R$ $(|z| \le P)$ with the usual topology ([1,2,6]). Instead of $\mathfrak{B}_n$ consider the spaces

$$\mathfrak{B}_r = \left\{ f(z);\ f(z) = \sum_n a_n z^n,\ \|f\|_r = \sum_n |a_n| r^n < \infty \right\}.$$

Then

$$\mathfrak{A}_R = \lim_{r \uparrow R} \operatorname{pr} \mathfrak{B}_r = \lim_{r_n \uparrow R} \operatorname{pr} \mathfrak{B}_{r_n}, \quad \overline{\mathfrak{A}}_R = \lim_{r \downarrow R} \operatorname{ind} \mathfrak{B}_r = \lim_{r_n \downarrow R} \operatorname{ind} \mathfrak{B}_{r_n}.$$

To each continuous linear operator A we associate a matrix $[a_{ik}]$:

$$Az^k = \sum_{i=0}^{\infty} a_{ik} z^i \quad (k = 0, 1, ...).$$

Theorem 7. Let a sequence of linear operators $A_n = [a_{ik}^{(n)}]$ $(n = 1, 2, ...)$ be given, continuously mapping: 1) $\mathfrak{A}_R$ into $\mathfrak{A}_P$; 2) $\mathfrak{A}_R$ into $\overline{\mathfrak{A}}_P$; 3) $\overline{\mathfrak{A}}_R$ into $\overline{\mathfrak{A}}_P$; 4) $\overline{\mathfrak{A}}_R$ into $\mathfrak{A}_P$. Then, in order that $\{A_n f\}$ converge for every element of the corresponding space, it is necessary and sufficient that the following conditions hold: a)

$$\lim_{n\to\infty} a_{ik}^{(n)} = a_{ik} \neq \infty \quad (i, k = 0, 1, ...);$$

b)

$$|a_{ik}^{(n)}| \leq Cr^k \rho^{-i} \quad (i, k = 0, 1, ... ; \ n = 1, 2, ...)$$

respectively: 1) for every $\rho < P$, $r = r(\rho) < R$ and $C = C(\rho) > 0$; 2) for some ρ, r and C, $\rho > P$, $r < R$, $C > 0$; 3) for every $r > R$, $\rho = \rho(r) > P$ and $C = C(r) > 0$; 4) for all ρ, r, $\rho < P$, $r > R$ and $C = C(r, \rho) > 0$.

The convergence conditions for a sequence of matrices can also be given in the terms used by M. G. Khaslanov to describe matrices continuously transforming analytic spaces in a disk ([3,5,6]).

G. Köthe gave a description of integral kernels $a(z', \zeta)$ representing continuous linear mappings of the space $\mathfrak{A}(G)$ into $\mathfrak{A}(G')$ and of $\mathfrak{A}(F)$ into $\mathfrak{A}(F')$, where G (G') are open, and F (F') closed subsets of the Riemann sphere Ω ([10]).

Denote by G_n $(G_{n'})$ domains approximating from within the open domain G (G') or from outside the closed domain F (F'), and by $\mathfrak{B}(G_n)$ $(\mathfrak{B}(G_{n'}))$ Banach spaces which, in their totality, define the analytic spaces (see ([10,6])).

Theorem 8. Let $\{A_k\}$ $(k = 1, 2, ...)$ be a sequence of linear operators continuously mapping: 1) $\mathfrak{A}(G)$ into $\mathfrak{A}(G')$; 2) $\mathfrak{A}(G)$ into $\mathfrak{A}(F')$; 3) $\mathfrak{A}(F)$ into $\mathfrak{A}(F')$; 4) $\mathfrak{A}(F)$ into $\mathfrak{A}(G')$, and

$$a_k(z', \zeta) = A_k \frac{1}{z - \zeta}$$

the integral kernels corresponding to these operators. Then, in order that $\{A_k f\}$ converge for every $f(z)$ of the corresponding space, it is necessary and sufficient that: 1) for every m there exist $n = n(m)$ such that $a_k(z', \zeta)$ $(k = 1, 2, ...)$ are locally analytic in the domain

$$\bigcup_m G'_m \times (\Omega \setminus G_{n(m)}),$$

$$\sup_k |a_k(z', \zeta)| = N_m < \infty \quad ((z', \zeta) \in G'_m \times (\Omega \setminus G_{n(m)+1})),$$

and for every $\zeta \in \Omega \setminus G$ the functions $a_k(z', \zeta)$ converge in $\mathfrak{A}(G')$ as $k \to \infty$; 2) for some m and n, $a_k(z', \zeta)$ $(k = 1, 2, ...)$ lo-

locally analytic in the domain $G'_m \times (\Omega \setminus G_n)$, $\sup_k |a_k(z', \zeta)| = N < \infty$ $((z', \zeta) \in G'_m \times (\Omega \setminus G_{n+1}))$, and, for any $\zeta \in \Omega \setminus G$, $a_k(z', \zeta)$ converges in $\mathfrak{A}(F')$ as $k \to \infty$; 3) for each n there exists $m = m(n)$ such that $a_k(z', \zeta)$ $(k = 1, 2, ...)$ are locally analytic in the domain

$$\bigcup_n G'_{m(n)} \times (\Omega \setminus G_n), \quad \sup_k |a_k(z', \zeta)| = N_n < \infty \quad ((z', \zeta) \in G'_{m(n)} \times (\Omega \setminus G_{n+1});\ n \geq 1),$$

and, for any $\zeta \in \Omega \setminus F$, $a_k(z', \zeta)$ converges in $\mathfrak{A}(F')$ as $k \to \infty$; 4) for arbitrary m and n, $a_k(z', \zeta)$ are locally analytic in the domain

$$\bigcup_{n,m} G'_m \times (\Omega \setminus G_n), \quad \sup_k |a_k(z', \Omega)| = N_{n,m} < \infty \quad ((z', \zeta) \in G'_m \times \Omega \setminus G_{n+1}),$$

$m, n = 1, 2, \ldots$, and, for any $\zeta \in \Omega \setminus F$, $a_k(z', \zeta)$ converges in $\mathfrak{A}(G')$ as $k \to \infty$.

As an application, let us consider the question of convergence in an analytic space in a disk of the series

$$Ay = \sum_{k=0}^{\infty} a_k(z) y^{(k)}(z) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k(z) y^{(k)}(z) = \lim_{n\to\infty} A_n y \tag{1}$$

for all elements $y(z)$ of the same space. In the case where the $a_k(z)$ are polynomials whose degrees are bounded, such conditions were obtained by Yu. F. Korobeinik ([11]).

Let

$$a_k(z) = \sum_i a_{ik} z^i$$

and let A correspond to the matrix $[\tilde{a}_{ik}]$,

$$\tilde{a}_{ik} = \sum_{0 \le j \le i,k} a_{i-j,k-j} \frac{k!}{j!} \qquad (i, n = 0, 1, \ldots).$$

Theorem 9. Let $a_k(z) \in \mathfrak{A}_R$, $k = 0, 1, \ldots$. In order that the series (1) converge in $\mathfrak{A}_{R_1}$, $R_1 \le R$, for all $y(z) \in \mathfrak{A}_R$, it is necessary and sufficient that, for each $\rho < R_1$, there exist $r = r(\rho) < R$ and $C = C(\rho) > 0$ such that

$$\left| \sum_{0, k-n \le j \le i,k} a_{i-j,k-j} \frac{k!}{j!} \right| \le C \frac{r^k}{\rho^i} \qquad (i, k = 0, 1, \ldots;\ n = 1, \ldots, k). \tag{2}$$

Theorem 10. Let $a_k(z) \in \mathfrak{A}_R$ $(k = 0, 1, \ldots)$. In order that the series (1) converge in $\overline{\mathfrak{A}}_{R_1}$, $R_1 < R$, for all $y(z) \in \mathfrak{A}_R$, it is necessary and sufficient that (2) hold for some $\rho \ge R_1$ and some $r < R$.

Theorem 11. Let $a_k(z) \in \overline{\mathfrak{A}}_R$ $(k = 0, 1, \ldots)$. In order that the series (1) converge in $\overline{\mathfrak{A}}_{R_1}$, $R_1 \le R$, for all $y(z) \in \mathfrak{A}_R$, it is necessary and sufficient that (2) hold for each $r > R$, with $\rho = \rho(r) > R_1$ and $C = C(r) > 0$.

Theorem 12. Let $a_k(z) \in \mathfrak{A}_R$, $k = 0, 1, \ldots$. In order that the series (1) converge in $\mathfrak{A}_{R_1}$, $R_1 \le R$, for all $y(z) \in \overline{\mathfrak{A}}_R$, it is necessary and sufficient that, for all $r > R$, $\rho < R_1$ and $C = C(\rho, r) > 0$, (2) hold.

Remark. The continuity of the matrix $[\tilde{a}_{ik}]$ is a necessary condition for convergence of the series (1), but not sufficient, as the example

$$Ay(z) = \sum_n (-1)^n \frac{z^n}{n!} y^{(n)}(z) = y(0).$$

Denote by $\Omega(\alpha, \delta)$ the angular domain

$$\{z;\ \alpha \leq \arg z \leq \alpha + \pi - \delta\},$$

where $0 \leq \alpha < 2\pi,\ 0 < \delta \leq \pi$.

Theorem 13. *If the coefficients $a_{ik} \in \Omega(a_{k-i}, \delta)$ $(i, k = 0, 1, ...)$, then in order that the series (1) converge for all elements of an analytic space in the topology of another analytic space in the disk (with the natural subordination of the radii of the disks), it is necessary and sufficient that the matrix $[\tilde{a}_{ik}]$ continuously map the first space into the second.*

Let $A_n = [a_{ik}^{(n)}]$ be a sequence of linear continuous operators mapping $\mathfrak{A}_R$ $(\overline{\mathfrak{A}}_R)$ into the space $\mathfrak{A}_P$ $(\overline{\mathfrak{A}}_P)$, and let Φ, $\Phi = \{\varphi_i\}$,

$$\Phi\left(\sum_n a_n z^n\right) = \sum_n a_n \varphi_n,$$

be some linear continuous functional on the space $\mathfrak{A}_P$ $(\overline{\mathfrak{A}}_P)$ (2).

Theorem 14. *In order that the sequence $\Phi(A_n f)$ converge for all $f \in \mathfrak{A}_R$ $(f \in \overline{\mathfrak{A}}_R)$, it is necessary and sufficient that, for some r, $r < R$ (for all r, $r > R$), the following hold: 1) for every k there exists the finite limit of the sequence*

$$\left\{\sum_j a_{jk}^{(n)} \varphi_j\right\}_{(n)};$$

2)

$$\overline{\lim}_{k\to\infty}\left(\sup_n \left|\sum_j a_{jk}^{(n)}\varphi_j\right|\right)^{1/k} < R \quad \left(\overline{\lim}_{k\to\infty}\left(\sup_n \left|\sum_j a_{jk}^{(n)}\varphi_j\right|\right)^{1/k} \leq R\right).$$

Theorem 15. *For the convergence of $\sum_k a_k y^{(k)}(z)$, where a_k are constants, at the point $z = z_0$, $|z_0| < R$ $(|z_0| \leq R)$, for all $y(z) \in \mathfrak{A}_R$ $(y(z) \in \overline{\mathfrak{A}}_R)$, it is necessary and sufficient that*

$$\overline{\lim}_{k\to\infty}\left(k! \sup_{0\leq n\leq k}\left|\sum_{k-n\leq j\leq n} a_{k-j}\frac{z_0^j}{j!}\right|\right)^{1/k} < R,$$

$$\times\left(\overline{\lim}_{k\to\infty}\left(k! \sup_{0\leq n\leq k}\left|\sum_{k-n\leq j\leq n} a_{k-j}\frac{z_0^j}{j!}\right|\right)^{1/k} \leq R\right).$$

Let us note that the proposed method is applicable to the study of analogous questions for generalized differential operators of infinite order and for functions of several variables.

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Note: Figure translations are in progress. See original paper for figures.