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Abstract

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MATHEMATICS

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ON THE REPRESENTATION OF LARGE NUMBERS BY TERNARY QUADRATIC FORMS

(Presented by Academician I. M. Vinogradov on 30 VII 1969)

Let $f(x_1, x_2, x_3)$ be a nondegenerate integral quadratic form. It is known ^(1,2) that the question of the asymptotic distribution of integer points on the surface

$$f(x_1, x_2, x_3) = n \tag{1}$$

as $n \rightarrow \infty$ is closely connected with the question of the boundary of the zeros of Dirichlet L -functions with real characters. In this note we present conditional theorems on the uniform distribution of integer points on the surface (1) in the case when f is a diagonal form.

Throughout what follows, unless the contrary is stated, the following is assumed to hold.

Assumption. There exists a number $\varepsilon > 0$ such that, for sufficiently large integral $|m|$, the function

$$L(s, \chi) = \sum_{d=1}^{\infty} \frac{\left(\frac{m}{d}\right)}{d^s} \quad (\operatorname{Re} s > 1)$$

has no zeros in the rectangle

$$1 - \frac{1}{(\ln |m|)^{1-\varepsilon}} < \operatorname{Re} s \leq 1, \quad |\operatorname{Im} s| \leq 1.$$

We shall use the following notation: $f(x_1, x_2, x_3)$ is a diagonal integral quadratic form of determinant D_f , whose coefficients are pairwise relatively prime and at least one of them is positive; $n > 0$ is an integer relatively prime to D_f and such that the congruence

$$f(x_1, x_2, x_3) \equiv n \pmod{8}$$

is primitively solvable; the function $L(s, \chi)$ is defined for $\operatorname{Re} s > 1$ by the series

$$L(s, \chi) = \sum_{d=1}^{\infty} \frac{\left(\frac{-nD_f}{d}\right)}{d^s};$$

Ω is an arbitrary finite convex domain on the surface

$$f(x_1, x_2, x_3) = 1, \quad (2)$$

$Q(n, \Omega)$ is the number of integer points in the projection of Ω onto the surface with equation (1), $\omega(\Omega)$ is the volume of the cone with base Ω and vertex at O ; if f is a positive form, then $Q_f(n)$ is the number of integer points on the ellipsoid (1); η_0 is a positive constant depending only on D_f and ε .

Theorem 1. *Let $f(x_1, x_2, x_3)$ be a positive form; then*

$$Q(n, \Omega) = c_0 \omega(\Omega) Q_f(n) + O\left(\frac{n^{1/2} L(1, \chi)}{(\ln n)^{\gamma_{10}}}\right);$$

c_0 and the constant entering into the remainder do not depend on Ω .

The method of proof of this theorem is analogous to the method proposed by Yu. V. Linnik (see ⁽¹⁾, p. 189) for studying the distribution of integral points on the sphere by segments; however, instead of ergodic considerations we use the scheme of work ⁽³⁾.

Yu. V. Linnik ⁽⁴⁾ and A. V. Malyshev (see ⁽²⁾, Ch. V), with the aid of the arithmetic of quaternions, proved a number of theorems on the representation of numbers by a given positive ternary quadratic form. From A. V. Malyshev's results it follows, in particular, that for a form f with odd coefficients, for some constant $\varkappa > 0$,

$$Q_f(n) > \varkappa n^{1/2} L(1, \chi),$$

if $L(s, \chi)$ has no zeros in the circle

$$|s - 1| < \frac{(\ln \ln n)^2 \ln \ln \ln n}{(\ln n)^{1/2}}.$$

Asymptotic formulas for the number of solutions of equation (1) are known only for a comparatively narrow class of positive forms (see ⁽²⁾, Ch. VI). Here we indicate one more class of forms for which one can find an asymptotic formula for $Q_f(n)$.

Theorem 2. Let $f(x_1, x_2, x_3)$ be a positive form such that, if p and q are arbitrary prime divisors of D_f , then $p \equiv 1 \pmod{4}$ and $(p/q) = 1$; then

$$Q_f(n) = \prod_{p|D_f} \frac{1}{p+1} Q(nD_f) + O\left(\frac{Q(nD_f)}{(\ln n)^{\gamma_{10}}}\right),$$

where $Q(nD_f)$ is the number of integral points on the sphere $x^2 + y^2 + z^2 = nD_f$.

In the works of Yu. V. Linnik ⁽⁵⁾ and B. F. Skubenko ⁽⁶⁾, the asymptotic-geometric properties of integral points on the surface (1) were studied in the two most important cases of the indefinite form f , when $D_f = \pm 1$, connected with the theory of reduction of binary quadratic forms. The distribution of integral points on hyperboloids of general type has not hitherto been considered.

Theorem 3. Let $f(x_1, x_2, x_3)$ be an indefinite form,

$$f(x_1, x_2, x_3) = a_1x_1^2 + a_2x_2^2 - a_3x_3^2,$$

where $a_2, a_3 > 0$ and $|D_f|$ is not a perfect square; suppose further that Ω_0, Ω are two regions on the hyperboloid (2), not containing points of the form $(x_1, x_2, 0)$, if $a_1 > 0$, and of the form $(0, x_2, 0)$ in the opposite case; then

$$Q(nD) = \frac{\omega(\Omega)}{\omega(\Omega_0)} Q(n, \Omega_0) + O\left(\frac{n^{1/2}L(1, \chi)}{(\ln n)^{\gamma_{10}}}\right).$$

The proof of this theorem differs little from the proof of Theorem 1.

Theorem 4. Let

$$f(x_1, x_2, x_3) = x_1^2 - x_2^2 + Dx_3^2,$$

where $D \neq 0$ is an arbitrary integer; Ω_0 is the region on the hyperboloid (2) which is defined by the inequalities

$$1 \leq x_1 - x_2 < 2,$$

then

$$2(x_1 - x_2) \leq x_3 < 4(x_1 - x_2);$$

where

$$Q(n, \Omega_0) = n^{1/2}g(n) \left(1 + O\left(\frac{1}{n^{\xi_0}}\right)\right),$$

where

$$g(n) \geq cL(1, \chi),$$

ξ_0 and c are some positive constants.

This theorem is unconditional. Its proof relies on the technique of the work ⁷, with the application of results of Burgess ⁸.

In conclusion I express my gratitude to A. I. Vinogradov for his attention to this work and for his advice.

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Note: Figure translations are in progress. See original paper for figures.

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