# AN OSCILLATOR ON AN ELASTIC-PLASTIC ELEMENT

MATHEMATICS

1970

SovietRxiv

## Abstract

## Full Text

UDC 517.9+539.214

*MATHEMATICS*

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# AN OSCILLATOR ON AN ELASTIC-PLASTIC ELEMENT

*(Presented by Academician A. Yu. Ishlinskii on 23 IV 1969)*

**1.** In the present paper we study the equation of oscillations of a material point on a one-dimensional elastic-plastic element.

Let $x(t)$ denote the variable coordinate of the oscillating point $N$ of mass $m$. Suppose that at time $t$ the following forces act on the point $N$: a variable external force $f(t)$, the friction force, the stress of an elastic element (which is determined by Hooke' s law), and the stress $\Phi$ of the elastic-plastic element. The equation of motion of the point $N$ is then written in the form

$$m\, d^2x/dt^2 + b\, dx/dt + Ex + \Phi = f(t). \tag{1}$$

We shall use the generalized Prager-Ishlinskii model of an elastic-plastic body (below the notation and concepts of ([1]) are used). Then the stress of the elastic-plastic element is determined by the formula

$$\Phi = \int_M F\{\alpha, \Gamma[x, l_-(\alpha), l_+(\alpha), l_0(\alpha)](t)\}\, d\mu(\alpha), \tag{2}$$

where $\Gamma[x, l_-, l_+, l_0]$ is the hysteresis operator; $F(\alpha, l)$ is a function that characterizes the stress of an infinitesimal element of the elastic-plastic body, and $\mu$ is a certain measure (describing the distribution of the elastic modulus with respect to the parameter $\alpha$).

We restrict ourselves to an analysis of the motion of the point $N$ under conditions in which the state $l(\alpha) = 0$ corresponds to the unstressed state of the elastic-plastic body; we shall assume that, within the limits of admissible extensions and compressions, the infinitesimal elements of the elastic-plastic body obey Hooke' s law. Then

$$\Phi = \int_M \Gamma[x, l_-(\alpha), l_+(\alpha), l_0(\alpha)](t)\, d\mu(\alpha). \tag{3}$$

The theorems presented below remain valid also in the case when $\Phi$ is determined by the general formula (2), provided that the nonlinear function $F(\alpha, l)$ satisfies a Lipschitz condition in the variable $l$.

Thus, we shall be interested in solutions of the equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + Ex + \int\limits_M \Gamma[x, l_-(\alpha), l_+(\alpha), l_0(\alpha)](t)\, d\mu(\alpha) = f(t). \tag{4}$$

**2. Theorem 1.** *Let the total modulus of elasticity of the plastic element be finite:*

$$\mu(M) = \int\limits_M d\mu(\alpha) < \infty. \tag{5}$$

*Then equation (4), for any summable initial state $l_0(\alpha)$ and for any initial condition*

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0 \tag{6}$$

*has a unique solution defined on the interval* $[0, \infty)$.

The proof of this theorem is easy to obtain by means of the contraction mapping principle, if one passes to the corresponding integral equation and then uses, on the one hand, the fact that the hysteron satisfies the Lipschitz condition, and, on the other hand, the easily proved inequality

$$|\, l_0(\alpha) - \Gamma[x, l_-(\alpha), l_+(\alpha), l_0(\alpha)](t)\,| \le \max_{0\le\tau;\, s\le t} |x(\tau) - x(s)|.$$

3. A function $x(t)$ $(0 \le t < \infty)$ is called asymptotically periodic with period $\omega$ if there exists an $\omega$-periodic continuous function $y(t)$ such that

$$\lim_{t\to\infty} |x(t) - y(t)| = 0. \tag{7}$$

A function $x(t)$ is called asymptotically almost periodic if from every sequence $h_n$ of positive numbers one can choose a subsequence $h_{n(i)}$ such that the sequence of functions $x(t + h_{n(i)})$ $(0 \le t < \infty)$ converges uniformly on the interval $[0, \infty)$. The asymptotically almost periodic functions $x(t)$, in the uniform norm

$$\|x(t)\| = \sup_{0\le t<\infty} |x(t)| \tag{8}$$

form a Banach space $B$. The asymptotically periodic functions with period $\omega$ form a subspace $B_\omega$ of the space $B$.

**Theorem 2.** *Suppose condition (5) is satisfied and suppose the initial state* $l_0(\alpha)$ *is summable.*

*Then the nonlinear operator*

$$\Pi x(t) = \int_M \Gamma[x, l_-(\alpha), l_+(\alpha), l_0(\alpha)](t)\, d\mu(\alpha) \tag{9}$$

*maps the space* $B$ *into itself and satisfies on it the Lipschitz condition with constant* $2\mu(M)$*:*

$$\|\Pi x - \Pi y\| \leq 2\mu(M)\|x - y\| \qquad (x, y \in B).$$

**Theorem 3.** *Suppose the conditions of Theorem 2 are satisfied.*

*Then the operator (9) leaves invariant each subspace* $B_\omega \subset B$ *of asymptotically periodic functions with period* $\omega$*.*

4. The last two theorems make it possible to give a general description of the oscillations of an oscillator on an elastoplastic element in the case of periodic, asymptotically periodic, almost periodic, and asymptotically almost periodic external forces $f(t)$. Such a description has been obtained under the assumption that the total modulus of elasticity $\mu(M)$ is not too large. In the theorems given below, as above, it is assumed that the initial state $l_0(\alpha)$ is summable.

**Theorem 4.** *Suppose the external force* $f(t)$ *is asymptotically periodic with period* $\omega$*.*

*Then, for sufficiently small total moduli of elasticity* $\mu(M)$*, every solution of equation (4) is an asymptotically periodic function with the same period* $\omega$*.*

**Theorem 5.** *Suppose the external force* $f(t)$ *is asymptotically almost periodic.*

*Then, for sufficiently small total moduli of elasticity* $\mu(M)$*, every solution of equation (4) is an asymptotically almost periodic function.*

In the case of an ordinary oscillator (a linear oscillator on a purely elastic element), in the presence of friction, the oscillations, as is well known, are globally asymptotically stable. The natural question arises as to whether

whether the same phenomenon occurs in the case of an oscillator on an elastic-plastic element. The answer is negative. There is only ordinary Lyapunov stability, both with respect to perturbations of the initial data and with respect to perturbations of the initial states of the plastic element. Let us give one assertion as an example.

**Theorem 6.** *Let the perturbing force* $f(t)$ *be bounded on the interval* $[0, \infty)$*.*

*Then there exists a* $\mu_0 > 0$ *such that, for* $\mu(M) < \mu_0$*, every solution of equation (4) is bounded and Lyapunov stable.*

**5.** Let us dwell in more detail on the case in which the perturbing force $f(t)$ is periodic. By analogy with nonlinear ordinary differential equations (and in connection with Theorem 4), one might expect that, for any fixed initial state $l_0(\alpha)$ of the elastic-plastic element, one can find initial conditions under which the oscillator executes periodic oscillations with the same period. It turns out that this hypothesis is false. However, for some initial states, one can specify initial conditions to which periodic solutions correspond.

**Theorem 7.** *Let an initial state $l_0(\alpha)$ of the elastic-plastic element and an $\omega$-periodic perturbing force $f(t)$ be given. Let $x(t)$ be an asymptotically periodic solution of equation (4) with period $\omega$. Let $y(t)$ be an $\omega$-periodic function such that*

$$\lim_{t\to\infty} |x(t) - y(t)| = 0.$$

*Then there exists an initial state $l_0^*(\alpha)$ of the elastic-plastic body for which the oscillator executes $\omega$-periodic oscillations $y(t)$.*

The initial state $l_0^*(\alpha)$ under the conditions of Theorem 3 is constructed quite simply. First the function

$$l(a;t) = \Gamma[x, l_-(a), l_+(a), l_0(a)](t)$$

is determined. This function turns out to be asymptotically periodic in the variable $t$ with period $\omega$. Therefore the sequence of functions $l(a, n\omega)$ $(n = 1, 2, ...)$ converges. The function $l_0^*(a)$ must be defined by the equality

$$l_0^*(\alpha) = \lim_{n\to\infty} l(\alpha, n\omega).$$

**6.** The theorems presented above are the simplest examples of assertions in the qualitative theory of differential-hysteretic equations. Apparently, many other theorems of the qualitative theory of ordinary differential equations can be extended to such equations. Differential-hysteretic equations are a special class of equations with Volterra operators; however, their theory should turn out to be substantially less complicated, since the hysteresis operator possesses (see (1)) a number of important additional properties.

We note that the numerical solution of differential equations with hysteretic terms is hardly more complicated than the numerical integration of ordinary differential equations.

Let us also point out that the functions $l_-(\alpha)$, $l_+(\alpha)$, and the measure $\mu(M)$, which determine the plastic body, can be determined experimentally.

The authors express their gratitude to B. M. Darinskii and A. Yu. Ishlinskii for discussion of the results obtained.

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Received
22 IV 1969

## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*