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Abstract

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ON THE HADAMARD WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR AN EVOLUTION EQUATION

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The uniformly well-posed Cauchy problem for an evolution equation in a Banach space has been well studied (see ([1])). If one considers systems of partial differential equations with constant coefficients of the form $\partial u/\partial t = A(D)u$, then necessary and sufficient conditions for the uniform well-posedness of the Cauchy problem for them in the space $\mathcal{L}_2(R^n)$ were obtained by H. O. Kreiss ([2]) and are complicated and difficult to verify. On the other hand, if a system is Petrovskii correct, i.e., if all eigenvalues of its characteristic matrix lie in some fixed half-plane, then, as I. G. Petrovskii showed ([3]), the Cauchy problem for it is Hadamard well-posed. Thus, Hadamard well-posed Cauchy problems form a considerably broader class than uniformly well-posed ones.

In the present paper a definition is given of Hadamard well-posedness of the abstract Cauchy problem for an evolution equation in a Banach space, and necessary and sufficient conditions for Hadamard well-posedness are found.

In a Banach space E with norm $\|\cdot\|$, consider the differential equation

$$dx/dt = Ax(t) \qquad (0 \le t < \infty) \tag{1}$$

with a linear closed operator A, having an everywhere dense domain of definition $D(A)$ in E.

A solution of equation (1) on the half-axis $[0, \infty)$ is a function $x(t)$ with values in $D(A)$, strongly continuously differentiable on $[0, \infty)$ and satisfying equation (1) for all $t \in [0, \infty)$. By the Cauchy problem is meant the problem of finding a solution on $[0, \infty)$ satisfying the initial condition

$$x(0) = x_0 \in D(A). \tag{2}$$

Let in the space E there be given a linear set E_1, contained in $D(A)$, on which another norm $\|\cdot\|_1$ and a seminorm $\|\cdot\|_2$ are defined.

Definition 1. The Cauchy problem (1), (2) is **Hadamard well-posed with type ω on the set E_1** if there exist constants C_1 and C_2 such that for any $x_0 \in E_1$ there exists a solution $x(t)$ satisfying the conditions

$$\|x(t)\| \leq C_1 e^{\omega t}\|x_0\|_1, \qquad \|Ax(t)\| \leq C_2 e^{\omega t}\|x_0\|_2,$$

and this solution is unique.

If the Cauchy problem is Hadamard well-posed, then by a change of the unknown function it is reduced to the same problem with type $\omega = 0$. In what follows we shall consider Hadamard well-posed problems for which

$$\|x(t)\| \leq C_1\|x_0\|_1, \qquad \|Ax(t)\| \leq C_2\|x_0\|_2 \quad (x_0 \in E_1). \tag{3}$$

We make the main assumption.

I. The positive half-axis does not belong to the point spectrum of the operator A, i.e., for $\lambda > 0$ there exists the operator $J_\lambda = (A - \lambda I)^{-1}$ (generally speaking, unbounded).

Lemma 1. *Under assumption I, for a given $x_0 \in D(A)$ there cannot exist more than one solution of problem (1), (2), bounded on the half-axis.*

Corollary. *Under assumption I, the uniqueness requirement in the definition of Hadamard well-posedness may be discarded.*

Definition 2. The maximal domain of Hadamard well-posedness is the linear set D of all elements x_0 for which there exists a solution of problem (1), (2), bounded together with its derivative on the half-axis $[0, \infty)$.

This definition is justified by the following consideration: by virtue of Lemma 1, on D the norm and seminorm are uniquely determined:

$$\|x_0\|_I = \sup_{0 \leq t < \infty} \|x(t)\|, \qquad \|x_0\|_{II} = \sup_{0 \leq t < \infty} \|Ax(t)\|,$$

where $x(t)$ is the solution referred to in Definition 2. It is obvious that, in the norms $\|\cdot\|_I$ and $\|\cdot\|_{II}$, problem (1), (2) is Hadamard well-posed on D. Furthermore, if problem (1), (2) is Hadamard well-posed with type $\omega = 0$ on some set E_1, then $E_1 \subset D$ and $\|x\|_I \leq C_1\|x\|_1$ and $\|x\|_{II} \leq C_2\|x\|_2$.

Under assumption I, let us study the properties of the operators $J_\nu (\nu > 0)$.

1°. The operator $J_\nu (\nu > 0)$ is defined on D, and the formula

$$J_\nu x_0 = -\int_0^\infty e^{-\nu\tau} x(\tau)\, d\tau \qquad (x_0 \in D,\ \nu > 0), \tag{4}$$

holds, where $x(t)$ is the corresponding solution of problem (1), (2).

2°. The set D is invariant with respect to the operators $J_\nu(\nu > 0)$, and the solution $y(t)$ of equation (1) with the initial condition $y(0) = J_\nu x_0$ is determined by the formula

$$y(t) = -\int_0^\infty e^{-\nu\tau} x(t+\tau)\, d\tau. \tag{5}$$

Moreover,

$$\|y(t)\| \le \frac{1}{\nu}\|x_0\|_I, \qquad \|Ay(t)\| \le \frac{1}{\nu}\|x_0\|_{II}.$$

3°. Hilbert' s identity holds:

$$J_\nu J_\lambda x_0 = \frac{1}{\lambda - \nu}(J_\lambda - J_\nu)x_0 \qquad (x_0 \in D;\ \lambda, \nu > 0),$$

as do the formulas for derivatives

$$J_\lambda^n x_0 = \frac{1}{(n-1)!}\frac{d^{n-1} J_\lambda x_0}{d\lambda^{n-1}} \qquad (x_0 \in D,\ \lambda > 0,\ n \ge 1). \tag{6}$$

4°. For $x_0 \in D$, $\lambda > 0$, and $n \ge 0$, the inequalities

$$\|J_\lambda^n x_0\| \le \lambda^{-n}\|x_0\|_I, \qquad \|J_\lambda^n A x_0\| \le \lambda^{-n}\|x_0\|_{II} \tag{7}$$

hold.

5°. The equalities (cf. (5))

$$\|x_0\|_I = \sup_{n\ge0}\sup_{\lambda>0}\|\lambda^n J_\lambda^n x_0\|, \tag{8}$$

$$\|x_0\|_{II} = \sup_{n\ge0}\sup_{\lambda>0}\|\lambda^n J_\lambda^n A x_0\| \tag{9}$$

hold.

From formula (5) it follows that the solution $z(t)$ of equation (1) with initial value $z(0) = \nu J_\nu x_0 + x_0$, for $x_0 \in D(A)$ and $\nu > 0$, is representable in the form

$$z(t) = -\int_0^\infty e^{-\nu\tau} A x(t+\tau)\, d\tau.$$

Hence, from (7) it follows

6°. The inequality holds

$$\|J_\lambda^n(\nu J_\nu x_0 + x_0)\| \le \frac{C_2}{\nu\lambda^n}\|x_0\|_1, \qquad x_0 \in D.$$

7°. For $x_0 \in D$ the relation

$$\lim_{\nu\to\infty}\|\lambda^n J_\lambda^n(\nu J_\nu A x_0 + A x_0)\| = 0 \tag{10}$$

holds uniformly with respect to $\lambda > 0$ and $n \geq 0$.

Let us introduce the set G, consisting of all elements $y \in E$ for which

$$\|y\|_G = \sup_{n\geq 0} \sup_{\lambda>0} \|\lambda^n J_\lambda^n y\| < \infty. \tag{11}$$

From the closedness of the operators J_λ it follows that G, with respect to $\|\cdot\|_G$, is a Banach space. From 1°, 2°, 4°, and 5° it follows that $D \subset G$ and $\|x\|_1 = \|x\|_G$ for $x \in D$. For every $y \in G$ the function $J_\nu y$ can be analytically continued by means of the expansion

$$J_\nu y = \sum_{k=0}^{\infty} (\nu - \lambda)^k J_\lambda^{k+1} y \qquad (\lambda > 0), \tag{12}$$

which converges in the disk $|\nu - \lambda| < \lambda$. Varying λ, we obtain an analytic continuation of $J_\nu x$ into the right half-plane.

The following assertion is essential:

Lemma 2. *The operator J_μ $(\mu > 0)$ is a bounded operator on G in the norm (11), and, moreover, for it the relation*

$$\|J_\mu x\|_G \leq \frac{1}{\mu}\|x\|_G \tag{13}$$

holds.

For the proof, the quantity $\|\lambda^n J_\lambda^n(\mu J_\mu x)\|$ for $\mu < \lambda$ can be estimated with the aid of the expansion (12):

$$\|\lambda^n J_\lambda^n(\mu J_\mu x)\| \leq \frac{\mu}{\lambda}\sum_{k=0}^{\infty}\left(1 - \frac{\mu}{\lambda}\right)^k \|\lambda^{n+k+1} J_\lambda^{n+k+1} x\| \leq \|x\|_G.$$

For $\lambda < \nu$ the same estimate can be obtained by expanding the function $\lambda^n J_\lambda^n x$ in a series at the point $\lambda = \mu$. From these estimates the inequality (13) follows.

Let F now denote the collection of all elements of G for which

$$\|\mu J_\mu y + y\|_G \to 0 \quad \text{as } \mu \to \infty. \tag{14}$$

By virtue of the uniform boundedness of the operators μJ_μ, F is a closed subspace of G. From the commutativity of the operators J_ν and J_μ, and the boundedness of J_ν in G, it follows that $J_\nu F \subset F$. The operators J_ν bounded in F form a pseudo-resolvent, and since they vanish only at zero, they are the resolvent of some operator $\widehat{A}$. By virtue of (14), the domain of definition of this operator is dense in F. It is not difficult to verify that the operator $\widehat{A}$ is the restriction of the operator A to the space F, i.e., it is defined on all elements $x_0 \in F \cap D(A)$ for which $Ax_0 \in F$, and on them coincides with A.

By what has been said and by Lemma 2, for the operator $\widehat{A}$ the conditions of the Hille-Yosida theorem are satisfied, and consequently it is the infinitesimal generator of a contraction semigroup $U(t)$ of operators in F (see $(1, 4)$). For every $x_0 \in D(\widehat{A})$ there exists a solution $x(t) = U(t)x_0$ of the equation

$$x' = \widehat{A}x$$

in F, satisfying the condition $x(0) = x_0$. This solution will also be

is a solution of problem (1), (2), since the norm in F is stronger than the norm in E. Further:

$$\sup_{0 \le t < \infty} \|x(t)\| \le \sup_{0 \le t < \infty} \|U(t)x_0\|_G = \|x_0\|_G < \infty,$$

$$\sup_{0 \le t < \infty} \|Ax(t)\| \le \sup_{0 \le t < \infty} \|U(t)\hat{A}x_0\|_G = \|Ax_0\|_G < \infty,$$

i.e. $x_0 \in D$.

Thus, $D(\hat{A}) \subset D$. On the other hand, by properties 1^0–7^0, $D \subset D(\hat{A})$, whence $D = D(\hat{A})$.

From all that has been said the following main results follow.

Theorem 1. *Under assumption I the maximal domain of Hadamard correctness consists of all elements x_0 from $D(A)$ possessing the properties:*

1) $x_0 \in G$ and $\|\mu J_\mu x_0 + x_0\|_G \to 0$ as $\mu \to \infty$;
2) $Ax_0 \in G$ and $\|\mu J_\mu Ax_0 + Ax_0\|_G \to 0$ as $\mu \to \infty$.

Theorem 2. *Let I be satisfied. In order that the Cauchy problem (1), (2) be Hadamard correct on a set E_1 with type $\omega = 0$, the following conditions are necessary and sufficient:*

1) for each $x_0 \in E_1$ all operators J_λ $(\lambda > 0)$ and all their possible products are defined;
2) the inequalities are valid $(x_0 \in E_1)$:

$$\|J_\lambda^n x_0\| \le C_1 \lambda^{-n} \|x_0\|, \qquad \|J_\lambda^n Ax_0\| \le C_2 \lambda^{-n} \|x_0\|_2;$$

3) uniformly with respect to $n \ge 0$ and $\lambda > 0$ the relations hold

$$\lim_{\mu \to \infty} \|\lambda^n J_\lambda^n (\mu J_\mu x_0 + x_0)\| = 0,$$

$$\lim_{\mu \to \infty} \|\lambda^n J_\lambda^n (\mu J_\mu Ax_0 + Ax_0)\| = 0.$$

Theorem 3. *Let I be satisfied and let the Cauchy problem be Hadamard correct with type $\omega = 0$ on the set E_1. Then there exists a space F, algebraically and topologically embedded in the space E and containing the set E_1, such that for the equation*

$$dx/dt = \hat{A}x,$$

where $\hat{A}$ is the restriction of A in the space F, the Cauchy problem is uniformly correct and has type $\omega = 0$.

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Note: Figure translations are in progress. See original paper for figures.