CRITERIA FOR NORMAL SOLVABILITY OF SYSTEMS OF SINGULAR INTEGRAL EQUATIONS AND WIENER-HOPF EQUATIONS

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Abstract

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Yu. LAITERER

CRITERIA FOR NORMAL SOLVABILITY OF SYSTEMS OF SINGULAR INTEGRAL EQUATIONS AND WIENER-HOPF EQUATIONS

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1. Let Γ be the unit circle and let L_2^k $(k = 1, 2, ...)$ be the Hilbert space of vector functions $j(\zeta) = j\{f_j(\zeta)\}_{j=1}^k$ with coordinates from $L_2(\Gamma)$ and norm

$$\|f\| = \left(\sum_{j=1}^{k} \|f_j\|_{L_2}^2 \right)^{1/2}.$$

In the present note we consider the singular integral operator T from $\mathfrak{R}(L_2^n, L_2^m)^*$, defined by the equality

$$(Tf)(\zeta) = c(\zeta)f(\zeta) + \frac{d(\zeta)}{\pi i} \int_\Gamma \frac{f(z)}{z-\zeta}\, dz \qquad (f \in L_2^n), \tag{1}$$

where $c(\zeta)$, $d(\zeta)$ are $m \times n$ matrices with elements continuous on Γ.

If $m = n$, then, as is known (^{1}), the operator T is a Φ-operator if and only if the rank of each of the matrices

$$a(\zeta) = c(\zeta) + d(\zeta), \qquad b(\zeta) = c(\zeta) - d(\zeta)$$

is equal to n for all $\zeta \in \Gamma$. If, however, for some $\zeta \in \Gamma$ at least one of the functions $\det a(\zeta)$ and $\det b(\zeta)$ vanishes, then, as shown in (^{2}), the operator T is neither a Φ_+- nor a Φ_--operator**.

In the present note necessary and sufficient conditions are established under which the operator T is normally solvable (Theorem 1). A necessary condition for normal solvability is the independence of the ranks of the matrices $a(\zeta)$ and $b(\zeta)$ from the argument ζ. It turns out that in the case $m > 1$ and $n > 1$ this condition alone is insufficient.

The results obtained are applied to Wiener-Hopf equations (Theorem 3). For a single equation ($n = m = 1$) the results of this note are set out in detail in (^{3}).

2. For all k denote by P the orthogonal projector in L_2^k, defined by the formula

$$(Pf)(\zeta) = \frac{f(\zeta)}{2} + \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{z-\zeta}\,dz \qquad (f \in L_2^k)$$

and put $Q = I - P$.

In what follows it will be convenient for us to write the operator T, defined by equality (1), in the form $T = aP + bQ$, where

$$a(\zeta) = c(\zeta) + d(\zeta) \quad \text{and} \quad b(\zeta) = c(\zeta) - d(\zeta).$$

Let $\operatorname{Ker} T = T^{-1}(0)$, and let $(\operatorname{Ker} T)^\perp$ be the orthogonal complement of the space $\operatorname{Ker} T$ in L_2^n. For $x \in \operatorname{Ker} T$ put

$$\|x\|_T = \rho(Px, \operatorname{Ker} aI) + \rho(Qx, \operatorname{Ker} bI).$$

* If $\mathfrak{B}_1, \mathfrak{B}_2$ are two Banach spaces, then by $\mathfrak{R}(\mathfrak{B}_1, \mathfrak{B}_2)$ we denote the set of all linear bounded operators acting from $\mathfrak{B}_1$ into $\mathfrak{B}_2$.

** An operator A is called normally solvable if its range is closed. It is called a Φ_+-(Φ_--)operator if, in addition, $\dim \operatorname{Ker} A < \infty$ ($\dim \operatorname{Coker} A < \infty$). If A is simultaneously a Φ_+- and a Φ_--operator, then it is called a Φ-operator.

It is easy to see that $\| \ \|_T$ is a norm on the linear manifold $\operatorname{Ker} T^\perp$.

Theorem 1. Let $a(\zeta)$, $b(\zeta)$ ($\zeta \in \Gamma$) be two continuous $m \times n$ matrix-functions.

In order that the operator

$$T = aP + bQ$$

from $\mathfrak{R}(L_2^n, L_2^m)$ be normally solvable, it is necessary and sufficient that the following two conditions be satisfied:

a) the rank of each of the matrices $a(\zeta)$ and $b(\zeta)$ does not depend on the parameter ζ on the unit circle;

b) $\operatorname{Ker} T^\perp$ is a complete Banach space with norm $\| \ \|_T$.*

It is not difficult to establish that condition b) is equivalent to the condition

$$\text{b'} \qquad \inf_{x \in \operatorname{Ker} T^\perp,\ \|x\|=1} \|x\|_T > 0.$$

Conditions a) and b) are independent. Indeed, put

$$a(\zeta) = \begin{pmatrix} \psi(\zeta) & 1 \\ \psi^2(\zeta) & \psi(\zeta) \end{pmatrix}, \qquad b(\zeta) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\psi(\zeta)$ is a continuous function on Γ that vanishes on an arc of the circle. It is easily shown that in this case condition a) is satisfied, whereas condition b) is not satisfied.

We note that, in general, condition b) is not satisfied when $\operatorname{Ker} T = \{0\}$, while at least one of the subspaces $\operatorname{Ker} aI$, $\operatorname{Ker} bI$ has positive dimension.

In the case where $m = n = 1$, the function $a(\zeta)$ vanishes at only one point and $b(\zeta) \equiv 1$, it is easily verified that condition b) is satisfied, whereas condition a) is not satisfied.

In the proof of the sufficiency of conditions a) and b) the complete continuity of the operators QaP and PbQ is used essentially (see (4)). In addition, the following assertion is used: if the rank of the matrix $a(\zeta)$ does not change on the unit circle, then the operator aI from $\mathfrak{R}(L_2^n, L_2^m)$ is normally solvable (see (5)).

The proof of the necessity of condition a) is the most difficult part of the proof. It is based on the following two lemmas:

Lemma 1. Let $a^*(\zeta)$ $(\zeta \in \Gamma)$ be a continuous matrix-function whose rank assumes different values on Γ.

Then there exists a constant matrix v of order m such that the matrix-function

$$f(\zeta) = \left\| f_{jk}(\zeta) \right\|_{\substack{j=1,\dots,n \\ k=1,\dots,m}} \overset{\text{def}}{=} a^*(\zeta) v$$

satisfies the following condition.

For every positive integer s there exist an interval $U_s \subseteq \Gamma$, a linear combination

$$(h_{s1}(\zeta), \dots, h_{sm}(\zeta)) = \sum_{j=1}^{n} \lambda_{sj} (f_{j1}(\zeta), \dots, f_{jm}(\zeta))$$

of rows of the matrix-function $f(\zeta)$, complex numbers $\beta_{s1}, \dots, \beta_{sm}$, and a positive number δ_s such that

$$\sum_{j=1}^{m} |\beta_{sj}|^2 = 1,$$

and the following estimates hold:

$$\sup_{\zeta\in U_s,\ \sum|\xi|^2=1}\left|\sum_{k=1}^{m} h_{sk}(\zeta)\xi_k\right|\le 2\delta_s,$$

$$\inf_{\zeta\in U_s}\left|\sum_{k=1}^{m} h_{sk}(\zeta)\beta_{sk}\right|\ge \tfrac{1}{2}\delta_s,$$

$$\sup_{\zeta\in U_s}\left|f(\zeta)\begin{pmatrix}\beta_{s1}\\ \cdot\\ \cdot\\ \cdot\\ \beta_{sm}\end{pmatrix}\right|_{n,2}\le\frac{1}{s},$$

$$\sup_{s,k}\|h_{sk}\|_{L_2(\Gamma)}<\infty,$$

where

$$|(\alpha_1,\dots,\alpha_n)|_{n,2}=\left(\sum_{j=1}^{n}|\alpha_j|^2\right)^{1/2}.$$

Lemma 2. Let R be an orthogonal projector in $L_2(\Gamma)$, and let φ,ψ be two functions from $L_2(\Gamma)$.

If $R\varphi=R\psi$, then for any measurable set $U\subseteq\Gamma$ the inequality

$$\|\varphi\|\,\|R\varphi\|_{\Gamma\setminus U}+(\|\psi\|_U+\|R\|_{\varphi U})\|R\varphi\|_U\ge 0$$

holds, where

$$\|\chi\|_V\overset{\text{def}}{=}\left(\int_V|\chi(\zeta)|^2\,|d\zeta|\right)^{1/2}$$

for $\chi\in L_2(\Gamma)$ and $V\subseteq\Gamma$.

We give an outline of the proof of the necessity of condition a). Suppose that the rank of the matrix $a(\zeta)$ on Γ changes. To prove that the operator T is not normally solvable, it is enough to construct a sequence of functions $y_s\in L_2^m$ for which the numbers

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* In the case $m=n=1$, condition b) is a consequence of condition a).

$$\alpha_s = \inf_x \|x\| / \|T^* y_s\|$$

increase without bound, where the lower bound is taken over all vectors $x \in L_2^n$ having the property $T^* x = T^* y_s$ $(T^* = Pa^* + Qb^*)$.

It turns out that such a sequence of functions may be the sequence

$$y_s(\zeta) = \zeta^{x_s} (\operatorname{mes} U_s)^{-1/2} \chi_{U_s}(\zeta)\, v \begin{pmatrix} \beta_{s1} \\ \cdot \\ \cdot \\ \cdot \\ \beta_{sm} \end{pmatrix},$$

where χ_{U_s} is the characteristic function of the interval U_s, and x_s is a sufficiently large integer.

The last assertion is proved with the help of the estimates from Lemma 1 and Lemma 2.

Theorem 2. If for all $\zeta \in \Gamma$ the ranks of the matrices $a(\zeta)$ and $b(\zeta)$ are equal to $\min\{m, n\}$, then the operator T is normally solvable.

Theorem 2 follows from Theorem 1.

3. From Theorems 1 and 2 one can derive criteria for the normal solvability of systems of equations of Wiener-Hopf type. Below we give examples of such theorems.

Let l_2^k be the Hilbert space of all sequences

$$x = \{(\xi_j^1, \dots, \xi_j^k)\}_{j=-\infty}^{\infty}$$

for which

$$\|x\|^2 = \sum_{s=1}^{k} \sum_{j=-\infty}^{\infty} |\xi_j^s|^2 < \infty,$$

and let $\tilde{l}_2^k$ be its subspace of all sequences of the form

$$\{(\xi_j^1, \dots, \xi_j^k)\}_{j=0}^{\infty} \quad (k = 1, 2, \dots).$$

If $\varphi(\zeta)$ is a continuous function on Γ, then by T_φ we denote the operator from $\mathfrak{R}(l_2^1, \tilde{l}_2^1)$ defined by the Toeplitz matrix $\|\varphi_{j-k}\|_{j,k=1}^{\infty}$, where φ_j $(j = 0, \pm 1, \dots)$ are the Fourier coefficients of the function φ. By U_φ we denote the operator from $\mathfrak{R}(\tilde{l}_2^1, \tilde{l}_2^1)$ defined by the matrix $\|\varphi_{j-k}\|_{j,k=-\infty}^{\infty}$.

The minimal angle (see (6)) between two subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of a Banach space is called the angle $\varphi^{(m)}(\mathfrak{M},\mathfrak{N})$ $(0 \leq \varphi^{(m)} \leq \pi/2)$, defined by the equality

$$\sin\varphi^{(m)}(\mathfrak{M},\mathfrak{N}) = \min\{\rho(S(\mathfrak{M}),\mathfrak{N}), \rho(S(\mathfrak{N}),\mathfrak{M})\},$$

where $S(\mathfrak{M})$ is the unit sphere of the subspace $\mathfrak{M}$.

Theorem 3. Let

$$a(\xi) = a_{jk}(\xi)\big|_{\substack{j=1,\dots,m\\k=1,\dots,n}}$$

be a continuous $m \times n$ matrix-function on Γ; T_a be an operator from $\mathfrak{R}(l_2^n, l_2^m)$, defined by the matrix

$$\|T_{a_{jk}}\|_{\substack{j=1,\dots,m,\\k=1,\dots,n}}$$

and U_a an operator from $\mathfrak{R}(\tilde{l}_2^n, \tilde{l}_2^m)$, defined by the matrix

$$\|U_{a_{jk}}\|_{\substack{j=1,\dots,m.\\k=1,\dots,n}}$$

In order that the operator T_a be normally solvable, it is necessary and sufficient that the following conditions hold:

a) the rank of the matrix $a(\xi)$ on Γ does not change;

β)

$$\sin\varphi^{(m)}(\operatorname{Ker} T_a^{\perp}, \operatorname{Ker} U_a) > 0,$$

where $\operatorname{Ker} T_a^{\perp}$ is the orthogonal complement of $\operatorname{Ker} T_a$ in l_2^n.

We note that conditions a) and β) are independent.

Theorem 4. Let T_a be the operator from the formulation of Theorem 3. If, for all $\xi \in \Gamma$, the rank of the matrix $a(\xi)$ is equal to $\min\{m,n\}$, then the operator T is normally solvable.

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Institute of Mathematics
Academy of Sciences of the MSSR
Kishinev

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Note: Figure translations are in progress. See original paper for figures.

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