ON BEST RATIONAL APPROXIMATIONS OF CONVEX FUNCTIONS AND FUNCTIONS OF BOUNDED VARIATION

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Abstract

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ON BEST RATIONAL APPROXIMATIONS OF CONVEX FUNCTIONS AND FUNCTIONS OF BOUNDED VARIATION

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Let $R_n[f]$ and $E_n[f]$ be the least deviations (in the metric $C[a,b]$) respectively of rational functions of degree not exceeding n and polynomials of degree not exceeding n from the function $f(x)$, $x \in [a,b]$. E. P. Dolzhenko ([1]) established that there exist continuous functions f for which $R_{n_k}[f] = E_{n_k}[f]$ for infinitely many indices n_k. At the same time f may have a modulus of continuity of any prescribed order of growth. In this connection there arises the problem of finding sufficiently simple classes of functions f for which $R_n[f]$ tends to zero essentially faster than $E_n[f]$.

We first dwell on convex functions. From a result of P. Szüsz and P. Turán ([2]) it follows: if $f(x)$ is convex on $[-1,1]$, then on any interval $[-1+\varepsilon, 1-\varepsilon]$ the estimate

$$R_n[f] \le C(\varepsilon) \ln^4 n/n^2 \quad (n = 2, 3, \ldots).$$

holds.

On the other hand (see ([3]), estimate (24)), there exist such piecewise-convex functions f for which $R_n[f]$ tends to zero arbitrarily slowly. In connection with these two results the question arises: do there exist convex functions which are approximated by rational functions arbitrarily badly? The answer to this is given by

Theorem 1. For an arbitrary continuous convex function $f(x)$ $(x \in [a,b], -\infty < a < b < \infty)$ the estimate

$$R_n[f] \le C_1 M \ln^2 n/n \quad (n = 2, 3, \ldots), \tag{1}$$

is valid, where C_1 is an absolute constant, and M is the maximum of $|f(x)|$ on $[a,b]$.

Theorem 2. There exists a convex continuous function $f(x)$, $x \in [0,1]$, for which the inequalities

$$R_n[f] \geq 1/n \ln^2 n \quad (n = 2, 3, \ldots).$$

hold.

Theorem 2 shows that, if factors of the type $\ln^\gamma n$ ($\gamma = \mathrm{const}$) are disregarded, estimate (1) is unimprovable.

We note in passing the estimate obtained by A. A. Gonchar (^{4}) (see there Corollary 3 of Theorem 1):

$$R_n\left[\left(\ln \frac{e}{x}\right)^{-\gamma}\right] \leq C_2 \left(\frac{\ln n}{n}\right)^\gamma, \quad \gamma > 0,\ x \in [0,1].$$

It follows from Theorem 1 that the last estimate (if the factor $\ln n$ is neglected) is not sharp for $0 < \gamma < 1$ (at the same time it is unimprovable for the piecewise-convex function $f_0(x)$: $f_0(x) = 0$ for $x \in [-1,0]$, $f_0(x) = (\ln e/x)^{-\gamma}$ for $x \in [0,1]$).

Theorem 2a. For any sequence $\varepsilon_1 \geq \varepsilon_2 \geq \cdots 0$, $\varepsilon_n \to 0$, there exists a subsequence of indices $n_1 < n_2 < \cdots$ and a convex non-

continuous function $f(x)$, $x \in [0,1]$, such that

$$R_{n_k}[f] > \varepsilon_{n_k}/n_k.$$

Theorem 3. If the convex function $f(x) \in \mathrm{Lip}\,\alpha$, $\alpha > 0$, $x \in [a,b]$, then the inequalities

$$R_n[f] \leq C_3 M \frac{\ln^6 n}{n^2} \qquad (n = 2, 3, \ldots), \tag{2}$$

hold, where C_3 depends only on α; M is the maximum of $|f(x)|$ on $[a,b]$.

Since D. Newman, P. Shapiro, and P. Turán [2] showed that there exists a convex function $f(x)$, $x \in [-1/2, 1/2]$, for which

$$R_n[f] > \frac{1}{n^2 \ln^2 n} \qquad (n = 2, 3, \ldots),$$

then (if one neglects factors of the type $\ln^\gamma n$, $\gamma = \mathrm{const}$) estimate (2) cannot be improved. The third theorem overlaps with the following result: for continuous functions of bounded variation $f(x) \in \mathrm{Lip}\,\alpha$, $x \in [a,b]$, $\alpha > 0$, the estimate* holds

$$R_n[f] \leq C_4 \frac{\ln^2 n}{n} \qquad (n = 2, 3, \ldots), \tag{3}$$

where C_4 does not depend on n. Both in estimate (2) and in estimate (3), on the right-hand side the Lipschitz exponent α enters only into the constants C_3 and C_4.

Theorem 4. Let $f(x)$ ($x \in [a, b]$) be a continuous function with modulus of continuity $\omega(\delta)$**, having finite total variation ($\operatorname{Var} f < \infty$). Then for $R_n = R_n[f]$ the estimate

$$\frac{1}{|\ln R_n|} \frac{R_n}{|\ln \omega^{-1}(R_n)|} \leq \frac{C_5}{n} \qquad (n \geq N(\omega)), \tag{4}$$

holds, where C_5 depends on $\operatorname{Var} f$ and on $\omega(\delta)$; $N(\omega)$ depends on ω; $\omega^{-1}(t)$ is the function inverse to $t = \omega(\delta)$.

Corollary. If $f(x)$, $x \in [a, b]$, is a continuous function of bounded variation whose modulus of continuity satisfies the inequality $\omega(\delta) \leq (\ln 1/\delta)^{-\gamma}$, $\gamma > 0$, then

$$R_n[f] \leq C_6 \left(\frac{\ln n}{n}\right)^{\gamma/(1+\gamma)} \qquad (n = 2, 3, \ldots), \tag{5}$$

where C_6 depends on $\operatorname{Var} f$ and on γ (we note that in the work [5] of G. Freud, for functions of this class the estimate $R_n[f] \leq C_7 n^{-\gamma/(2+\gamma)}$ was obtained).

The work was carried out under the supervision of E. P. Dolzhenko.

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CITED LITERATURE

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3. A. A. Gonchar, *Matem. sborn.*, **72** (114), no. 3, 489 (1967).

4. A. A. Gonchar, *Matem. sborn.*, **73** (115), no. 4, 630 (1967).

5. G. Freud, *Acta Math. Acad. Sci. Hung.*, **17** (3-4), 313 (1966).

* The result was obtained independently by A. A. Abugattarapov with E. P. Dolzhenko and G. Freud and was reported at the International Congress of Mathematicians in Moscow in 1966; see also (5).

** $\omega(\delta)$ is a nonnegative function, nondecreasing for $\delta \geq 0$, $\omega(0) = 0$. The inverse function $\omega^{-1}(t)$ is defined as follows: $\omega^{-1}(t) = \inf\{\delta : \omega(\delta) \geq t\}$.

Note: Figure translations are in progress. See original paper for figures.

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