A HOMEOMORPHISM THEOREM FOR GENERAL ELLIPTIC BOUNDARY-VALUE PROBLEMS WITH BOUNDARY CONDITIONS THAT ARE NOT NORMAL

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Abstract

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A HOMEOMORPHISM THEOREM FOR GENERAL ELLIPTIC BOUNDARY-VALUE PROBLEMS WITH BOUNDARY CONDITIONS THAT ARE NOT NORMAL

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Homeomorphism theorems for elliptic boundary-value problems were established in papers (^{1}–^{11}). In all these papers it is assumed that the boundary conditions are normal. Meanwhile, it is well known that the Noether property of elliptic problems holds without the assumption of normality of the boundary expressions. Therefore the problem naturally arises of establishing homeomorphism theorems without the assumption of normality of the boundary conditions. This problem was formulated in conversations with the author by Yu. M. Berezanskii and M. I. Vishik; it was also posed by Madzhennes in (^{5}). The present paper is devoted to the solution of the indicated problem; it is a continuation of (^{12}), whose basic notation and results are used here. We also note paper (^{13}), in which a homeomorphism theorem is established (without the assumption of normality of the boundary expressions) in spaces conjugate to the Hölder spaces.

1. In a bounded domain G with boundary Γ, consider a properly elliptic differential expression $L = L(x, D)$ of order $2m$ with complex coefficients, and on Γ a system of m expressions $B_j(x, D)$, in general pseudodifferential in the tangential directions and differential in the directions normal to Γ, of orders $m_j \leqslant 2m - 1$, which cover L. For simplicity it is assumed that both the surface Γ and all the expressions are infinitely smooth. It is well known that for every real $s \geqslant 0$ the operator

$$A_s : u \to (Lu, B_1 u|_\Gamma, \dots, B_{mu}|_\Gamma) \quad (u \in W_2^{2m+s}(G)) \tag{1}$$

is Noetherian

$$\text{from } W_2^{2m+s}(G) \text{ into } W_2^s(G) \dotplus \sum_{j=1}^{m} W_2^{2m+s-m_j-1/2}(\Gamma) \equiv K_s(G), \tag{2}$$

i.e. $\mathfrak{N} = \{u \in W_2^{2m+s}(G) : A_{su} = 0\}$ is finite-dimensional (and does not depend on s), the range $\mathfrak{R}(A_s)$ of the operator A_s is closed in $K_s(G)$ and has finite codimension. Moreover, there exists a finite-dimensional space $\mathfrak{N}^+ \subset C^\infty(\bar{G}) \dotplus C^{\infty,m}(\Gamma)$ independent of $s \geqslant 0$ $(C^{\infty,r}(\Gamma) = C^\infty(\Gamma) \dotplus \dots \dotplus C^\infty(\Gamma))$ such that $F = (f, \varphi_1, \dots, \varphi_m) \in K_s(G)$ belongs to $\mathfrak{R}(A_s)$ if and only if

$$[F, V] \equiv (f, v) + \sum_{j=1}^{m} \langle \varphi_j, v_j \rangle = 0 \quad (V = (v, v_1, \dots, v_m) \in \mathfrak{N}^+) \tag{3}$$

$((\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle$ denote the scalar product respectively in $L_2(G)$ and $L_2(\Gamma))$.

As in $(^{6-10,\ 12,\ 14})$, denote by $\widetilde{W}_2^l(G) = \widetilde{W}_{2,2m}^l(G)$ (l an arbitrary integer) the completion of the set $C^\infty(\bar{G})$ with respect to the norm

$$|||u|||_l = \left(\|u\|^2_{W_2^l(G)} + \sum_{j=1}^{2m} \|D_\nu^{j-1} u\|^2_{W_2^{l-j+1/2}(\Gamma)} \right)^{1/2} \tag{4}$$

$\Bigl(D_\nu = \dfrac{1}{\nu}\dfrac{\partial}{\partial \nu},\ \nu = \nu(x)$ is the unit vector of the inward normal to Γ at the point x;

the spaces $W_2^l(G)$ and $W_2^{-l}(G)$ $(W_2^{l-1/2}(\Gamma)$ and $W_2^{-(l-1/2)}(\Gamma))$ are mutually conjugate with respect to $(\cdot,\cdot)$ $(\langle\cdot,\cdot\rangle)$. If $l \geq 2m$, then the norms $|||u|||_l$ and $\|u\|_{W_2^l(G)}$ are equivalent and $\widetilde{W}_2^l(G) = W_2^l(G)$; for $l < 2m$ these norms are not equivalent.

Since the norm (4) is the norm of the direct sum

$$W_2^l(G) \dotplus \sum_{j=1}^{2m} W_2^{l-j+1/2}(\Gamma),$$

in essence $\widetilde{W}_2^l(G)$ consists of elements of the form $u = (u_0, u_1, \dots, u_{2m})$, where $u_0 \in W_2^l(G)$, $u_j = D_\nu^{j-1} u_0|_\Gamma$, if $l - j \geq 0$; when $l - j < 0$, u_j is an arbitrary element of the space $W_2^{l-j+1/2}(\Gamma)$. If t is not an integer, $l < t < l+1$, then define $W_2^t(G)$ by complex interpolation between the spaces $\widetilde{W}_2^l(G)$ and $\widetilde{W}_2^{l+1}(G)$. For arbitrary real t, the norm $\|u\|_{\widetilde{W}_2^t(G)}$ is equivalent to the norm $\|u\|_{W_2^t(G)} + \|Lu\|_{W_2^{t-2m}(G)}$ $(^{14})$. Below $u|_G$ is the first component of an element $u \in \widetilde{W}_2^t(G)$.

For each real t, the closure by continuity A_t of the mapping

$$u \to (Lu, B_1 u|_\Gamma, \dots, B_{mu}|_\Gamma) \qquad (u \in C^\infty(\bar{G}))$$

acts continuously from all of $\widetilde{W}_2^{2m+t}(G)$ into

$$K_t(G) = W_2^t(G) \dotplus \sum_{j=1}^{m} W_2^{2m+t-m_j-1/2}(\Gamma).$$

We now formulate the main result of this paper.

Theorem 1. *In order that the problem* $A_{tu} = F \in K_t(G)$ *have a solution* $u \in W_2^{2m+t}(G)$*, it is necessary and sufficient that the element* $F = (f, \varphi_1, \dots, \varphi_m)$ *satisfy relations (3). The restriction* $\widetilde{A}_t$ *of the operator* A_t *to the subspace*

$$\widetilde{P}\,\widetilde{W}_2^{2m+t}(G) = \{u \in \widetilde{W}_2^{2m+t}(G) : (u|_G, \mathfrak{N}) = 0\}$$

of the space $\widetilde{W}_2^{2m+t}(G)$ *realizes a homeomorphism*

$$\widetilde{P}\,\widetilde{W}_2^{2m+t}(G) \to \widetilde{Q}^+ K_t(G),$$

where

$$\widetilde{Q}^+ K_t(G) = \{F \in K_t(G) : [F, \mathfrak{N}^+] = 0\}$$

is a subspace of $K_t(G)$*.*

2. To prove Theorem 1, we first, using the Green formula derived in (12), describe the set $\mathfrak{N}^+$ more concretely. Denoting by

$$Bu = (B_1 u, \dots, B_{mu}), \qquad Cu = (C_1 u, \dots, C_{mu}),$$

$$B'v = (B'_1 v, \dots, B'_{mv}), \qquad C'v = (C'_1 v, \dots, C'_{mv}),$$

$$\xi^n = (u^n|_\Gamma, \dots, D_\nu^{2m-1} u^n|_\Gamma),$$

we write Green' s formula (23) from (12) in the form

$$(Lu, v) + \langle Bu, C'v\rangle_{L_2^m(\Gamma)} = (u, L^+ v) + \langle Cu, B'v\rangle_{L_2^m(\Gamma)} + \langle \xi^n, Tv\rangle_{L_2^{2m}(\Gamma)} \qquad (5)$$

$$(u, v \in W_2^{2m}(G)).$$

Let $\hat{r}$ and r be operators assigning to each $2m$-dimensional vector $\eta = (\eta_1, \dots, \eta_{2m})$, respectively, the vectors

$$\hat{r}\eta = (\eta_1, \dots, \eta_m) \quad \text{and} \quad r\eta = (\eta_{m+1}, \dots, \eta_{2m}).$$

Represent the space $\mathfrak{N}_\Gamma^+$ in the form

$$\mathfrak{N}_\Gamma^+ = \mathfrak{N}_\Gamma^{+1} \oplus \mathfrak{N}_\Gamma^{+2} \oplus \mathfrak{N}_\Gamma^{+3},$$

where

$$\mathfrak{N}_\Gamma^{+1} = \{\eta \in \mathfrak{N}_\Gamma^+ : r\eta = 0\}, \qquad \mathfrak{N}_\Gamma^{+2} = \{\eta \in \mathfrak{N}_\Gamma^+ : \hat{r}\eta = 0\};$$

if $\eta \neq 0$

belongs to $\mathfrak{N}_\Gamma^{+3}$, then, since $r\hat{\eta} \neq 0$, also $r\eta \neq 0$. Let

$$W_2^{2m+s}(\mathrm{pr}) = \{u \in W_2^{2m+s}(G) : Bu|_\Gamma = 0\} \quad (s \geq 0),$$

$$W_2^{2m}(\mathrm{pr})^+ = \{v \in W_2^{2m}(G) : (Lu, v) = (u, L^+v) \ (u \in W_2^{2m}(\mathrm{pr}))\}.$$

From Green' s formula it follows that

$$W_2^{2m}(\mathrm{pr})^+ = \{v \in W_2^{2m}(G) : \langle Tv, \mathfrak{N}_\Gamma \rangle_{L_2^{2m}(\Gamma)} = 0, \quad B'v|_\Gamma \in r\mathfrak{N}_\Gamma^{+3}\}. \tag{6}$$

We also put

$$W_2^{2m+s}(\mathrm{pr})^+ = W_2^{2m}(\mathrm{pr})^+ \cap W_2^{2m+s}(G) \quad (s \geq 0).$$

Consider problems with homogeneous boundary conditions

$$Lu = f \in W_2^s(G), \quad u \in W_2^{2m+s}(\mathrm{pr}) \quad (s \geq 0); \tag{7}$$

$$L^+v = g \in W_2^s(G), \quad v \in W_2^{2m+s}(\mathrm{pr})^+ \quad (s \geq 0). \tag{8}$$

Lemma 1. *The space $\mathfrak{N}^+$ of solutions of problem (8) with $g = 0$ is finite-dimensional and does not depend on $s \geq 0$. For solvability of problem (7) it is necessary and sufficient that $(f, \mathfrak{N}^+) = 0$, and for solvability of problem (8) it is necessary and sufficient that $(g, \mathfrak{N}) = 0$.*

Now consider problems with nonhomogeneous boundary conditions:

$$Lu = f, \quad Bu|_\Gamma = \varphi \quad (\varphi = (\varphi_1, \dots, \varphi_m)); \tag{9}$$

$$L^+v = g, \quad B'v|_\Gamma - \psi \in r\mathfrak{N}_\Gamma^{+3}, \quad \langle Tv, \mathfrak{N}_\Gamma \rangle_{L_2^{2m}(\Gamma)} = 0 \tag{10}$$
$$(\psi = (\psi_1, \dots, \psi_m)).$$

For any real t, for each vector

$$\varphi \in \sum_{j=1}^{m} W_2^{2m+t-m_j-1/2}(\Gamma)$$

we construct a vector $\tilde{\varphi} \in r\mathfrak{N}_\Gamma^{+3}$ such that the vector $(\varphi, \tilde{\varphi})$ is orthogonal in $L_2^{2m}(\Gamma)$ to $\mathfrak{N}_\Gamma^{+3}$. It is clear that by these conditions $\tilde{\varphi}$ is uniquely determined by the vector φ; if $e_1, \dots, e_q$ is a basis in $\mathfrak{N}_\Gamma^{+3}$ such that $re_1, \dots, re_q$ forms an orthonormal (with respect to $L_2^m(\Gamma)$) basis in $r\mathfrak{N}_\Gamma^{+3}$, then

$$\tilde{\varphi} = -\sum_{j=1}^{q} \langle \varphi, \hat{r}e_j \rangle_{L_2^m(\Gamma)} re_j. \tag{11}$$

Theorem 2. *In order that problem (9) with $F = (f, \varphi) \in K_s(G)$ $(s \geq 0)$ have a solution $u \in W_2^{2m+s}(G)$, it is necessary and sufficient that*

$$\langle \varphi, r\mathfrak{N}_\Gamma^{+1} \rangle_{L_2^m(\Gamma)} = 0; \tag{12}$$

$$(f, v) + \langle \varphi, C'v \rangle_{L_2^m(\Gamma)} - \langle \tilde{\varphi}, B'_{jv} \rangle_{L_2^m(\Gamma)} = 0 \quad (v \in \mathfrak{N}^+).$$

In order that problem (10) with

$$(g,\psi) \in W_2^s(G) + \sum_{j=1}^{m} W_2^{2m+s-m'_j-1/2}(\Gamma) \equiv K'_s(G)$$

$$(s \geq 0)$$

have a solution $v \in W_2^{2m+s}(G)$, *it is necessary and sufficient that*

$$\langle \psi, r\mathfrak{N}_\Gamma^{+2} \rangle_{L_2^m(\Gamma)} = 0, \quad (g,u) + \langle \psi, Cu \rangle_{L_2^m(\Gamma)} = 0 \quad (u \in \mathfrak{N}). \tag{13}$$

The necessity of this theorem follows from Green' s formula (5) and the considerations of Sec. 2 of [12]. Sufficiency is proved by reducing nonhomogeneous problems to homogeneous ones and using Lemma 1. We note also that conditions (12) (taking (11) into account) make explicit the solvability conditions (3) for problem (9).

Consider also the problem

$$Lu(x) = f(x) \quad (x \in G), \qquad Bu|_\Gamma = \varphi, \quad \langle Cu, \mathfrak{r}\mathfrak{N}_\Gamma^+ \rangle_{L_2^m(\Gamma)} = 0. \tag{14}$$

Theorem 3. In order that problem (14) with $F = (f,\varphi) \in K_s(G)$ $(s \geq 0)$ have a solution $u \in W_2^{2m+s}(G)$, it is necessary and sufficient that

$$\langle \varphi, \mathfrak{r}\mathfrak{N}_\Gamma^+ \rangle_{L_2^m(\Gamma)} = 0, \qquad (f,v) + \langle \varphi, C'v \rangle_{L_2^m(\Gamma)} = 0 \quad (v \in \mathfrak{N}^+). \tag{15}$$

From Theorems 2, 3 and Green' s formula (5), by means of the method of M. I. Vishik–S. L. Sobolev ([15]), it follows that in order that problem (14) with $F \in K_s(G)$ $(s \leq 0)$ have a solution $u \in \widetilde{W}_2^{(2m+s)}(G)$, it is necessary and sufficient that relations (15) be satisfied. Hence it already follows easily that, in order that problem (9) with $F \in K_s(G)$ $(s \leq 0)$ have a solution $u \in \widetilde{W}_2^{2m+s}(G)$, it is necessary and sufficient that relations (12) be satisfied, i.e. Theorem 1 follows.

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Note: Figure translations are in progress. See original paper for figures.

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