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ON THE HOMOTOPY INVARIANCE OF RATIONAL PONTRYAGIN NUMBERS

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The paper investigates the question of the homotopy invariance of the rational Pontryagin numbers of closed smooth ($PL-$) manifolds. Pontryagin numbers of the form

$$\langle L_k(M^n) y_1 \dots y_{n-4k}, [M^n] \rangle, \qquad \text{where } y_1, \dots, y_{n-4k} \in H^1(M^n).$$

are considered. For them homotopy invariance is proved (Theorem 1). Homotopy invariance of Pontryagin numbers of this kind was first studied in works ([4–6]). It turns out that no other relations of homotopy invariance of rational Pontryagin numbers exist for manifolds with free abelian fundamental group (see Part II).

I. **Theorem 1.** *Let M^n be a closed oriented smooth (or $PL-$) manifold of dimension n. Consider in the group $H_{n-1}(M^n)$ arbitrary elements $\xi_1, \dots, \xi_{n-4k}$. Then the scalar product $\langle L_k(M^n), \xi_1 \circ \dots \circ \xi_{n-4k} \rangle$ is homotopy invariant (here $L_k(M^n)$ is the Hirzebruch polynomial in the Pontryagin classes of the manifold M^n, and $\xi_1 \circ \dots \circ \xi_{n-4k}$ is the intersection of the cycles $\xi_1, \dots, \xi_{n-4k}$).*

Let $f : M^n \to N^n$ be a homotopy equivalence of degree $+1$ of connected closed oriented smooth manifolds of dimension n. Consider an indivisible element $z \in H_{n-1}(M^n)$ and the covering $\hat{M}_z$ over M^n, defined by the normal divisor $\pi \subset \pi_1(M^n)$, which contains exactly those elements of $\pi_1(M^n)$ whose image under the natural homomorphism $\pi_1(M^n) \to H_1(M^n)$ has zero intersection index with the cycle z. Over N^n consider the covering $\hat{N}_{f_*(z)}$, constructed analogously from the element $f_*(z) \in H_{n-1}(N^n)$. The covering map $\hat{M}_z \to \hat{N}_{f_*(z)}$ will be denoted by h. Let W^{n-1} be an arbitrary connected submanifold of N^n realizing the cycle $f_*(z)$. Obviously, there exists a covering embedding $W^{n-1} \subset \hat{N}_{f_*(z)}$.

Theorem 2. If $\pi_1(M^n)$ is free abelian, $\pi_1(W) \to \pi_1(\hat{N}_{f_*(z)})$ is a monomorphism, and $n \geq 6$, then there exists a map $h_0 : \hat{M}_z \to \hat{N}_{f_*(z)}$, homotopic to h, t-regular on W^{n-1} and inducing a homotopy equivalence of the submanifolds $V_0^{n-1} = h_0^{-1}(W^{n-1})$ and W^{n-1}.*

Remark 1. The proof of this theorem repeats the main stages of the proof of Theorem 3 of paper ([5]). The difference consists in the fact that, instead of the kernels of the maps in homology on the universal coverings under the embedding of the submanifold into the right and left parts of the encompassing open manifold, in the proof of Theorem 2 one uses, respectively, the kernels of the map h_* in homology on the universal coverings of the right and left parts modulo the dividing submanifold.

Remark 2. Theorem 2 remains valid also for manifolds with boundary; moreover, if $V^{n-1} = h^{-1}(W^{n-1})$ is such that on ∂V^{n-1} the map h is already a homotopy equivalence, then V_0^{n-1} can be chosen so that $\partial V_0^{n-1} = \partial V^{n-1}$.

* Proof correction note. As the author has learned, this theorem was announced in a significantly stronger for

Remark 3. By analogy with (3), one can realize V_0^{n-1} inside the manifold M^n itself, i.e., there exists a mapping $f_0 : M^n \to N^n$, homotopic to f, t-regular on W^{n-1} and inducing a homotopy equivalence of the submanifolds $V_0^{n-1} = f_0^{-1}(W^{n-1})$ and W^{n-1}.

First we derive Theorem 1 from Theorem 2. Let $f : M^n \to N^n$ be a homotopy equivalence of smooth manifolds (the case of PL-manifolds is completely analogous), $\xi_1, \dots, \xi_{n-4k} \in H_{n-1}(\dot{M}^n)$. Obviously, to prove the assertion of Theorem 1 it suffices to do so under the assumption that $\pi_1(M)$ is abelian, since the commutant can be killed by surgeries without destroying the homotopy equivalence. Passing to finite-sheeted coverings, we reduce the problem to the case of a free abelian fundamental group. We shall assume that $\xi_1 \circ \dots \circ \xi_{n-4k}$ is a free element in $H_{4k}(M)$, since otherwise the assertion of the theorem is trivial. The rest of the proof proceeds by induction. We indicate the first step. Replacing, if necessary, the element ξ_1 by an indivisible element z_1, we realize $f_*(z_1)$ by a submanifold $W^{n-1} \subset \hat{N}_{f_*(z_1)}$, where $\pi_1(W') \to \pi_1(\hat{N}_{f_*(z_1)})$ is a monomorphism. Applying Theorem 2, we obtain a homotopy equivalence $h_0 : V_0^{n-1} \to W^{n-1}$. Next we apply Theorem 2 to the cycle $\xi_2 \cap V_0^{n-1} = z_2 \in \hat{H}_{n-2}(V_0^{n-1})$, and so on. The argument does not go through for $k = 1$, when at the last step we obtain a homotopy equivalence $f : M^5 \to N^5$ and Theorem 2 is not applicable. In this case the assertion follows from Theorem 1 of (4).

We now give a sketch of the proof of Theorem 2. Denote $h^{-1}(W)$ by V. It is easy to show that W divides $\hat{N}_{f_*(z)}$ into two parts K and L. Hence V divides $\hat{M}_z$ into two parts $A = h^{-1}(K)$ and $B = h^{-1}(L)$. Denote the mapping $h|_V : V \to W$ by g. From the arguments of § 4 of (5) and from the finite generation of $H_1(\hat{N}_{f_*(z)}) = \pi_1(\hat{N}_{f_*(z)})$ it follows that $\pi_1(W) \to \pi_1(\hat{N}_{f_*(z)})$ is an isomorphism.

We shall successively perform surgeries on V^{n-1} inside $\hat{M}_z$, killing the kernels of the mapping g_* and changing the mapping h correspondingly at each surgery. In the surgeries only disks from the kernels of the mappings $h_* : \pi_i(A, V) \to$

$\pi_i(K, W)$ and $h_* : \pi_i(B, V) \to \pi_i(L, W)$ will be used. It is easy to make V connected. Further, using Browder' s construction (1), by surgeries of the indicated kind one can arrange that g_* becomes an isomorphism on π_1. We obtain

$$\pi_1(\hat{M}_z) = \pi_1(\hat{N}_{f_*(z)}) = \pi_1(V) = \pi_1(W) = \pi_1(A) = \pi_1(B) = \pi_1(K) = \pi_1(L).$$

Following (8), we shall denote the kernels of the mappings in i-dimensional homology on universal coverings by K_i, and the cokernels in compact cohomology by K_c^i. The groups K_i, K_c^i are modules over the group ring $\mathbb{Z}(\pi)$. In (8) it is proved that, for manifolds (with boundary and without boundary), K_i and K_c^i split off as direct summands in the corresponding homology and cohomology groups, and that all exact sequences of pairs and triples are preserved for them, and also Poincaré duality holds. We shall also need the exact sequence of the pair $(\hat{M}_z, V)$ for the groups K_i and some others, which are obtained in an entirely analogous way.

We have $K_i(\hat{M}_z) = 0$, $K_1(V) = 0$. From the exact sequence of the pair $(\hat{M}_z, V)$ we obtain $K_i(V) = K_{i+1}(\hat{M}_z, V)$. This gives the direct decomposition we need:

$$K_i(V) = K_{i+1}(A, V) \oplus K_{i+1}(B, V),$$

where $K_2(A, V) = K_2(B, V) = 0$. One can show (using the methods of (2)) that if $K_j(A, V) = 0$, $K_j(B, V) = 0$ for $j \le i$, then any element of $K_{i+1}(A, V)$ is realized by such a mapping of pairs $(D^{i+1}, S^i) \to (A, V)$ that the composite mapping

$$(D^{i+1}, S^i) \to (A, V) \to (K, W)$$

is 0 in the group $\pi_{i+1}(K, W)$. For $i \le [(n-1)/2]$ there exists a homotopic mapping which is a smooth embedding (the proof is analogous to the proof of Lemma 3.1 from (5), with the condition $\pi_2(A, V) = 0$ replaced by $K_2(A, V) = 0$). This permits one to perform all surgeries in dimensions $i < [(n-1)/2]$. If $n - 1 = 2m$, then from Poincaré duality we obtain that only $K_m(V) \ne 0$. Po-

therefore $K_m(V)$ is stably free over $Z(\pi)$, and hence the modules $K_{m+1}(A, V)$ and $K_{m+1}(B, V)$ are stably free ($\pi = Z + \dots + Z$).

For $n - 1 = 2m + 1$ we also paste $K_{m+1}(B, V)$. After this, only $K_{m+1}(A, V)$ and $K_{m+2}(B, V)$ will be different from 0. By analogy with the proof of Lemma 6.1 from ([5]) we prove the projectivity of these modules. Denote by T the generator of the group Z of motions of the covering $\hat{M}_z$. Then, for sufficiently large s, $T^{sA} \subset A$ and the homomorphisms of inclusion $K_m(B) \to K_m(T^{sB})$, $K_{m+1}(T^{sA}) \to K_{m+1}(A)$ are trivial. Denote $A \setminus T^{sA}$ by Q. Let $Q \cap T^{sA} = V'$, $Q \cap B = V$. From the exact sequences of triples $(\hat{M}_z, A, T^{sA})$, $(\hat{M}_z, T^{sB}, B)$ for the manifold Q we obtain that $K_i(Q, V) = 0$ ($i \ne m,\ m+1$), $K_m(Q, V) = K_{m+1}(Q, V) = K_{m+1}(A, V)$, and also $K_i(Q, V') = 0$ ($i \ne m+1,\ m+2$), $K_{m+1}(Q, V') = K_{m+2}(Q, V') = K_{m+2}(B, V)$. Hence, in complete analogy with § 6 of ([5]), we obtain the projectivity of $K_{m+1}(A, V)$ and $K_{m+2}(B, V)$. Now the kernels $K_{m+1}(A, V)$ for $n-1=2m$ and for $n-1=2m+1$ are annihilated in exactly the same way as in ([5]). We obtain a submanifold $V_0 \subset \hat{M}_z$ such that $K_i(V_0)=0$ for all i. Theorem 2 is proved.

II. We shall further assume that all manifolds under consideration have a free abelian fundamental group. In ([7]), homomorphisms $\alpha : L_n(\pi \times Z) \to L_{n-1}(\pi)$ and $\beta : \operatorname{Ker}\alpha \to L_n(\pi)$ were constructed in a purely geometric way, where $L_n(\pi)$ is Wall' s group of obstructions to surgeries for manifolds of dimension n with fundamental group π (see ([9])). Consider a map $f : (M^n, \nu) \to (X^n, \xi)$, where M^n and X^n are smooth manifolds, ξ is a stable bundle over X^n, ν is the stable normal bundle to M^n, $\deg f = +1$, $\pi_1(M^n) = \pi_1(X^n) = \pi \times Z$, $n \geq 6$. Then f determines a certain element $[f] \in L_n(\pi \times Z)$. Consider an arbitrary submanifold $Y^{n-1} \subset X^n$ such that $\pi_1(Y^{n-1}) = \pi \subset \pi_1(X^n)$ (in ([7]), $X^n = Y^{n-1} \times S^1$). Generalizing the construction ([7]) to the case under consideration, we define

$$\alpha([f]) = [f|_N : (N^{n-1}, \nu|_N) \to (Y^{n-1}, \xi|_Y)],$$

and also, in complete analogy with ([7]), the homomorphism β. Correctness ($\alpha(0)=0$, $\beta(0)=0$) is ensured by Theorem 2. One may make the supposition that α and β do not depend on the choice of X^n, i.e. are universal. The author does not have a proof of this fact; however, the universality of α and β entails interesting consequences, which we present here. These consequences were pointed out to the author by S. P. Novikov.

1. Let M^n be a closed smooth oriented manifold, $\pi_1(M^n)$ a free abelian group, η a stable bundle over M^n, and ν the stable normal bundle to M^n. If the bundle η satisfies the conditions: $J(\eta) = J(\nu)$ and $\langle L_k(\eta), \xi_1 \circ \cdots \circ \xi_{n-4k} \rangle = \langle L_k(\nu), \xi_1 \circ \cdots \circ \xi_{n-4k} \rangle$ for arbitrary $\xi_1, \dots, \xi_{n-4k} \in H_{n-1}(M^n)$, then the obstruction to the existence of a manifold N^n homotopy equivalent to M^n and having stable normal bundle η has finite order in the group $L_n(\pi_1(M^n))$.

2. For manifolds with free abelian fundamental group, the rational Pontryagin classes are not related by any other relations of homotopy invariance, except for the relations indicated in Theorem 1.

3. Let M^n be a closed smooth oriented manifold, $\pi_1(M^n)$ a free abelian group, and η a stable bundle over M^n. For the finiteness of the number of smooth manifolds of the homotopy type M^n with stable normal bundle η, it is necessary and sufficient that the natural mapping

$$\Lambda^{n-4q-1} H^1(M^n) \to H^{n-4q-1}(M^n)$$

be a monomorphism for every $q \geq 0$.

The proofs of these propositions follow from the following method of computing the obstruction lying in $L_n(Z^k)$. Let, as above, $f : (M^n, \nu) \to (X^n, \xi)$, $\pi_1(X) = Z^k$. Consider a free basis $\xi_1, \dots, \xi_k$ in the group $H_{n-1}(X^n)$. Let $\xi_{i_1} \circ \cdots \circ \xi_i$ be a free element in the group

$H_{n-q}(X^n)$ $(i_1 < i_2 < \dots < i_q)$. Represent it by such a submanifold $Y^{n-q} \subset X^n$ that $\pi_1(Y^{n-q}) \subset \pi_1(X^n)$. Then, considering $N^{n-q} = f^{-1}(Y^{n-q})$ and $f|_N$: $(N^{n-q}, \nu|_N) \to (Y^{n-q}, \xi|_Y)$, we obtain the obstruction $a^q([f]) = \alpha \circ \dots \circ \alpha([f]) \in L_{n-q}(Z^{k-q})$. Further, when $n - q \equiv 0 \pmod 4$ there exists a natural homomorphism $\gamma : L_{n-q}(Z^{k-q}) \to L_{n-q}(1)$—the signature of the obstruction matrix divided by 8. Thus, for the set $i_1 < \dots < i_q$, when $n - q \equiv 0 \pmod 4$, we obtain a certain number (an element of $L_{n-q}(1)$); moreover, the element $[f] \in L_n(Z^k)$ has finite order if and only if all the numbers thus obtained are equal to 0. The last assertion follows from the universality of α and β and from (7).

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Note: Figure translations are in progress. See original paper for figures.