ON THE TOPOLOGICAL TYPES OF DYADIC SPACES

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Abstract

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ON THE TOPOLOGICAL TYPES OF DYADIC SPACES

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It is said (([1]), vol. 1, p. 116) that spaces X and Y are spaces of one and the same topological type if X is homeomorphic to Y.

Main theorem. *Let τ be an infinite cardinal. Then: (I). There exist exactly* $\exp\tau$ *different topological types of dyadic bicompacts of weight τ—as many as there are all types of bicompacts of weight τ. (II). There exist exactly* $\exp\exp\tau$ *topological types of dyadic bicompacts of density τ—as many as there are all types of bicompacts of density τ. (III). There exist exactly* $\exp\exp\tau$ *topological types of closed and pseudocompact subsets of the generalized Cantor discontinuum D^τ, of weight $\tau \geq \aleph_1$—as many as there are all types of completely regular spaces of weight τ. (IV). There exist exactly* $\exp\exp\exp\tau$ *topological types of dyadic spaces of density τ—as many as there are all different types of completely regular spaces of density τ.*

This theorem is a generalization and strengthening of a number of results of Mazurkiewicz and Sierpiński ([2]), Mostowski (([3]), p. 44), M. Ya. Perelman ([4]), I. I. Parovichenko ([5]), and Reichbach ([6]) (an extensive bibliography is in ([1]), vol. 1). Let us note that all upper estimates are obtained easily. Indeed, by A. N. Tikhonov' s theorem, every completely regular space of weight τ is topologically contained in the Tikhonov cube I^τ (bicompacts–as closed subsets). Therefore the number of topological types of bicompacts of weight $\leq \tau$ does not exceed the cardinality of the set of all closed subsets of I^τ, i.e. $\exp\tau$, and the number of topological types of completely regular spaces does not exceed the cardinality of the set of all subsets of I^τ, i.e. $\exp\exp\tau$. Further, by a theorem of E. Čech ([6]), every bicompact X of density $\leq \tau$ is a continuous image of βT_τ, the Stone-Čech compactification of a discrete space of cardinality τ. Then $wX \leq w(\beta T_\tau) \leq \exp\tau$, and the preceding arguments can be applied. Finally, considering the compactification bX of an arbitrary completely regular space X of density $\leq \tau$, we obtain a bicompact bX of density $\leq \tau$. Then $wX \leq w(bX) \leq \exp\tau$. Hence the number of such spaces does not exceed the cardinality of the set of all subsets of the Tikhonov cube $I^{\exp\tau}$, i.e. $\exp\exp\exp\tau$. The main difficulty in proving such theorems is obtaining lower estimates, since in doing this one must "construct" or prove the existence of a very "large" set of different spaces with prescribed properties. Let us note that in the main theorem a fairly narrow class of spaces is considered—dyadic ones (for example, every space of this class satisfies Suslin's condition), while the cardinal estimates are the same as in the general situation.**

§ 1. **A method of constructing a "large" set of bi-**

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* A dyadic bicompact is a continuous image of D^τ; a dense subset of a dyadic bicompact is called a dyadic space in the sense of V. I. Ponomarev ([9]). For the basic properties of dyadicity see ([1]), vol. 2, or ([10]). By the density sX of a space X we mean the minimum cardinality of everywhere dense subsets in X. By $\exp\tau$ is denoted 2^τ, and wX is the weight of X.

** Item IV of the main theorem is astonishing. From it, for example, it follows that there exists a hyper-hypercontinuum of different types of separable dyadic spaces!

bicompacts.* Let ω_τ be the least ordinal of cardinality τ, and let γ_α be an arbitrary ordinal: $0 < \gamma_\alpha \le \omega_\tau$. Denote by $F(\gamma_\alpha)$ the set of all ordinals less than or equal to $\omega_\tau^{\gamma_\alpha}$, endowed with the interval topology. It is not hard to show that $F(\gamma_\alpha)$ is a zero-dimensional bicompactum of weight τ, and that $F(\gamma_\alpha)$ is homeomorphic to $F(\gamma_\beta)$ if and only if $\gamma_\alpha = \gamma_\beta$. The ordinal γ_α is called the order of the bicompactum $F(\gamma_\alpha)$, or the order of the point $\omega_\tau^{\gamma_\alpha}$, which is the last point in it.

Theorem 1. *There exist exactly* $\exp\tau$ *topological types of zero-dimensional bicompacta* $A(\xi)$ *of weight* τ, *each of which has the following structure: for every neighborhood* $U \subset A(\xi)$ *there exists a neighborhood* $V \subset U$ *which either is homeomorphic to the Cantor perfect set, or contains an isolated point.*

Proof. Let E be the set of all ordinals less than or equal to $\eta = \omega_\tau^\tau$. In the interval topology E is a zero-dimensional bicompactum of weight τ. Insert into each interval $(a, a+1)$ a Cantor set C so that the left endpoint of C coincides with a, and the right endpoint with $a+1$. Denote the resulting bicompactum by X. Consider the bicompactum

$$Z = (E \times E) \cup [(0, \eta) \times X].$$

For each point $a \in E$ put

$$S(a) = \{(x, y) \in Z,\ x = a\}.$$

It is clear that $S(a)$ is similar to E for all $a \in E$. Further, let T be the set of all non-limit ordinals in E. It is clear that T is discrete in E and $\overline{T} = E$. Let $\xi = \{\gamma_\alpha\}$, $\alpha < \omega_\tau$, be an arbitrary set of pairwise distinct ordinals such that $0 < \gamma_\alpha \le \omega_\tau$. It is easy to understand that the cardinality of the set of all such ξ is $\exp\tau$. For each $\gamma_\alpha \in \xi$ choose a well-ordered set $F(\gamma_\alpha) \subset S(\alpha)$, $\alpha \in T$, whose order type is $\omega_\tau^{\gamma_\alpha}+1$, and the point (α,η) has order γ_α relative to $F(\gamma_\alpha)$. This is possible, since $S(\alpha)$ is similar to E. Put

$$A(\xi)=\bigcup_{\alpha\in T}F(\gamma_\alpha)\ \cup\ \bigcup_{\alpha\in E-T}S(\alpha)\ \cup\ [(0,\eta)\times X].$$

Then $A(\xi)$, by virtue of the discreteness of T and the closedness of $F(\gamma_\alpha)$, for each $\alpha\in T$ is a closed subset of Z and hence a bicompactum. We shall prove that if $\xi_1\neq\xi_2$, then $A(\xi_1)$ is not homeomorphic to $A(\xi_2)$. Suppose that $A(\xi_1)$ is homeomorphic to $A(\xi_2)$, and at the same time there exists $\gamma_\alpha\in\xi_1$ such that $\gamma_\alpha\notin\xi_2$. Since under homeomorphisms the perfect part is carried into the perfect part, $(0,\eta)\times X\subset A(\xi_1)$ is carried into $(0,\eta)\times X\subset A(\xi_2)$. Since isolated points are close to points of E, while near points of $X-E$ there are no isolated points, $E\times(0,\eta)\subset A(\xi_1)$ is carried into $E\times(0,\eta)\subset A(\xi_2)$. Therefore $T\times(0,\eta)\subset A(\xi_1)$ is carried into $T\times(0,\eta)\subset A(\xi_2)$. This means that the point $(\alpha,\eta)\in A(\xi_1)$, having order γ_α relative to $F(\gamma_\alpha)$, $\alpha\in T$, is carried into some point $(\alpha',\eta)\in A(\xi_2)$, having the same order γ_α relative to $F(\gamma_{\alpha'})$. But this contradicts the fact that $\gamma_\alpha\notin\xi_2$. Thus, $(\xi_1\neq\xi_2)\iff(A(\xi_1)\neq A(\xi_2))$. This means that the cardinality of the set of all topological types of the form $A(\xi)$ is not less than the cardinality of the set of all sequences $\xi=\{\gamma_\alpha\}$, i.e. $\exp\tau$. It can be shown that each $A(\xi)$ possesses the properties we need. The theorem is proved.

§ 2. Dyadic bouquet. Let X and Y be topological spaces homeomorphic to D^τ, and let $f:A\to B$ be a homeomorphism, where $A\subset X$ and $B\subset Y$. Denote by $\mathfrak{D}(A)$ the space obtained by gluing the set X to the set Y by means of the mapping f. In other words, the elements of the decomposition are the individual points $x\in(X\oplus Y)-(A\oplus B)$ and the pairs $(x,f(x))$, if $x\in A$. The space $\mathfrak{D}(A)$ will be called a dyadic bouquet of weight τ over A. Obviously, $\mathfrak{D}(A)$ is a dyadic bicompactum if A is closed in X. Moreover, $\mathfrak{D}(A)$ is an irreducible image of D^τ if A is closed and nowhere dense in X. We note that the bouquet $\mathfrak{D}\{x\}=D^\tau\vee D^\tau$ was considered in ([3]), and the bouquet $\mathfrak{D}(aT)$, where aT is the Alexandroff compactification of a discrete space T of uncountable cardinality, was considered by Engelking ([11]). The character $\chi(A)$ of a space A is called–

* We somewhat modify and generalize the construction of Reichbach ([6]), which gives a continuum of metrizable compacta.

take $\sup\chi(x,A)$, where $\chi(x,A)$ is the character of the point x in A, i.e., the minimum of the cardinalities of fundamental systems of neighborhoods of the point x in A.

Lemma. Let $\mathfrak{D}(A)$ be a dyadic bouquet of weight $\tau\geq\aleph_1$ over A, and let $\chi(A)=\mathfrak{n}<\tau$. Then $\mathfrak{D}(A)$ is not homeomorphic to D^τ.

Proof. Suppose that $\mathfrak{D}(A)=D^\tau$. Then, by a theorem of one of the authors ([8]), every canonical closed subset of $\mathfrak{D}(A)$ has type G_δ. Hence $\overline{X}$ has type G_δ. Consequently, $V=\mathfrak{D}(A)-\overline{X}$ has type F_σ. On the other hand, $A=Y-V=D^\tau-V$. Thus, for every point $x\in A$ we have

$$\chi(x, D^\tau) \le \chi(x, A) \cdot \chi(A, D^\tau) \le \mathfrak{n}\aleph_0 = \mathfrak{n} < \tau.$$

This contradicts the fact that $\chi(x, D^\tau) = \tau$ for all $x \in D^\tau$ ([8]). The lemma is proved.

§ 3. **Proof of the main theorem.** (I). Without loss of generality assume that $\tau \ge \aleph_1$. Let $\Gamma = \{A_\xi\}$ be the family of cardinality $\exp\tau$ of pairwise non-homeomorphic zero-dimensional bicompacts of weight τ, constructed in § 1. By a theorem of N. B. Vedenisov ([10]), each A_ξ is topologically contained in D^τ as a closed subset. Consider the family $\mathfrak{D} = \{\mathfrak{D}(A_\xi)\}$ of dyadic bouquets of weight τ over A_ξ. We shall prove that $\mathfrak{D}(A)$ is not homeomorphic to $\mathfrak{D}(B)$ if A is not homeomorphic to B. From this, as is not hard to see, assertion (I) of our theorem will follow. Suppose the contrary. Let $f : \mathfrak{D}(A) \to \mathfrak{D}(B)$ be a homeomorphism. We shall show that $f(A) \subseteq B$. Suppose that there exists a point $x \in A$ such that $y = f(x) \in D^\tau - B$. This means that there exists a clopen neighborhood W of the point y homeomorphic to D^τ. Let U be a clopen neighborhood of the point x such that $f(U) \subset W$. Denote $U_0 = A \cap U$. By Theorem 1 there exists a neighborhood $V_0 \subset U_0$ which is either homeomorphic to $D^{\aleph_0}$, or contains an isolated point. (In the latter case, as V_0 we take this isolated point.) Let V be a clopen subset in $\mathfrak{D}(A)$, contained in U, and such that $V_0 = V \cap A$. Then V is homeomorphic either to $\mathfrak{D}(D^{\aleph_0})$ or to $\mathfrak{D}(\{x\})$. Consequently, by the lemma, V is not homeomorphic to D^τ ($\tau \ge \aleph_1$). On the other hand, since f is a homeomorphism, $f(V)$ is a clopen subset of $W = D^\tau$ and, consequently, $f(V)$ is homeomorphic to D^τ. A contradiction. Thus, $f(A) \subseteq B$. Similarly it is proved that $f^{-1}(B) \subseteq A$. Hence $f(A) = B$, contrary to the assumption. Thus (I) is proved.

(II). Put $\mathfrak{m} = \exp\tau$, $\tau \ge \aleph_0$. Then, by the theorem of Hewitt–Marczewski–Pondiczery ([10]),* we have $sD^{\mathfrak{m}} \le \tau$. Hence each of the bicompacts $\mathfrak{D}(A_\xi)$ constructed in (I), $A_\xi \subset D^{\mathfrak{m}}$, has weight $\mathfrak{m}$ and density $\le \tau$. The number of pairwise non-homeomorphic bicompacts of the form $\mathfrak{D}(A_\xi)$ of weight $\mathfrak{m}$ is equal to $\exp\mathfrak{m} = \exp\exp\tau$, as was required to prove.

(III). Let $\tau \ge \aleph_1$. Denote by Σ the set of all points of D^τ at which only a countable number of coordinates are different from zero. Note that Σ is a dense and pseudocompact subset of D^τ of weight τ, which is a Fréchet–Urysohn space (the sequential closure of any subset in Σ coincides with its topological closure). By virtue of the equality $\tau = 2\tau$ the space D^τ can be represented in the following form:

$$D^\tau = D^{\tau_1} \times D^{\tau_2} = \bigcup_{x \in D^{\tau_1}} M_x, \quad \text{where } M_x = D^{\tau_2} \text{ and } M_x \cap M_y = \emptyset, \text{ for } x \ne y, \tag{1}$$

where $\tau_1 = \tau_2 = \tau$. For $X \subset D^{\tau_1}$ put

$$F(X) = \bigcup_{x \in X} M_x \cup \Sigma$$

and denote by $\mathfrak{F}$ the family of sets $F(X)$, where X runs through the family of all subsets of the space D^{τ_1}. For each X the space $F(X)$ is a dense and pseudocompact subset of D^τ. Further, if $x_0 \in X - Y$, then $M_{x_0} \subset F(X)$. On the other hand, $M_{x_0} \not\subset F(Y)$, since $M_{x_0} \cap \bigcup_{x \in X} M_x = \emptyset$

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* An analogous theorem was proved in 1941 by M. Ya. Perelman ([4]).

and Σ contains no copy of $D^\tau = M_x$. (Recall that for $\tau \geq \aleph_1$ the space Σ is Fréchet-Urysohn, whereas D^τ is not.) Thus the cardinality of the family $\mathfrak{F}$ is equal to $\exp\exp\tau$. Next, observe that for each $F(X) \in \mathfrak{F}$ the cardinality of the set of all continuous maps of $F(X)$ into D^τ, which we denote by $\mathfrak{F}(X)$, is equal to $\exp\tau$. Indeed, by continuity, each map $f : F(X) \to D^\tau$ is uniquely determined by a map of some dense subset $M \subset F(X)$ of cardinality τ into D^τ. On the other hand, the cardinality of the set of all maps (not necessarily continuous) of M into D^τ does not exceed

$$(\operatorname{card} D^\tau)^{\operatorname{card} M} = (\exp\tau)^\tau = \exp\tau.$$

Thus, $\operatorname{card}\mathfrak{F} = \exp\exp\tau$ and $\operatorname{card}\mathfrak{F}(X) = \exp\tau$. It follows immediately that in D^τ there exist $\exp\exp\tau$ topological types of dense and pseudocompact subsets. Thus (III) is proved.

(IV). Put $\mathfrak{m} = \exp\tau$. Then $sD^\mathfrak{m} \leq \tau$. Let K be some dense subset of $D^\mathfrak{m}$ of cardinality τ. Represent $D^\mathfrak{m}$ in the form (1), with τ_1, τ_2, and τ replaced by $\mathfrak{m}_1, \mathfrak{m}_2$, and $\mathfrak{m}$. For $X \subset D^{\mathfrak{m}_1}$ put

$$F(X)_1 \cup \bigcup_{x \in X} M_x \cup K.$$

Again denote by $\mathfrak{F}$ the family of sets $F(X)$, where X runs through the family of all subsets of the space $D^{\mathfrak{m}_1}$. For each X the space $F(X)$ is a dense subset of $D^\mathfrak{m}$, and $sF(X) \leq \tau$. Since

$$\exp\mathfrak{m} = \operatorname{card} D^\mathfrak{m} > \operatorname{card} K = \tau,$$

we have $(F(X) = F(Y)) \Longleftrightarrow (X = Y)$. Thus

$$\operatorname{card}\mathfrak{F} = \exp\exp\mathfrak{m} = \exp\exp\exp\tau.$$

Next, arguing in the same way as in (III), we obtain that the cardinality of the set of all continuous maps of $F(X)$ into $D^\mathfrak{m}$ does not exceed

$$(\exp\mathfrak{m})^\tau = \exp\exp\tau$$

for each $F(X) \in \mathfrak{F}$. From this we immediately obtain that there exist $\exp\exp\exp\tau$ topological types of dense subsets of $D^{\mathfrak{m}}$, each of which has density $\leq \tau$. The main theorem is completely proved.

Corollary 1. For every cardinal $\tau \geq \aleph_0$ there exist exactly $\exp\tau$ topological types of connected irreducibly dyadic bicompacta of weight τ.

Corollary 2. For every cardinal $\tau \geq \aleph_1$ there exist exactly $\exp\tau$ algebraic types of dense (in the sense of (3)) subalgebras of the free Boolean algebra of cardinality τ.

Corollary 3. For every cardinal $\tau \geq \aleph_0$ there exist exactly $\exp\tau$ algebraic types of subalgebras of the form $C(X)$ of the Banach algebra $C(D^\tau)$.

Let us note that in cases (III) and (IV) we proved only the existence of a "large" set of topological types of a given weight or density. It would be interesting, with the help of ordinals, to "construct" the elements of this set, as was done in (I) and (II). Here are other questions to which the authors have not found answers:

1) How many topological types of locally connected dyadic bicompacta of weight τ exist?
2) How many topological types of open subsets of D^τ, $\tau \geq \aleph_1$, exist? (We note that in the case $\tau = \aleph_0$ there exist two types [7], despite the fact that there exist $\exp\aleph_0$ topological types of closed subsets.)
3) How many types of discontinuity ((1), vol. 1) of dyadic bicompacta of weight τ exist?

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REFERENCES

1. K. Kuratowski, *Topology*, 1, Moscow, 1966; 2, Moscow, 1969.
2. S. Mazurkiewicz, W. Sierpiński, *Fundam. Math.*, 1, 17 (1920).
3. P. Sikorski, *Boolean Algebras*, Moscow, 1969.
4. A. A. Markov, in: *Mathematics in the USSR over 30 Years*, Moscow, 1948, p. 183.
5. I. I. Parovichenko, *Dokl. AN SSSR*, 115, No. 5, 866 (1957).

6. J. Kelley, *General Topology*, Moscow, 1968.

7. M. Reichbach, *Proc. Am. Math. Soc.*, 13, No. 1, 17 (1962).

8. B. A. Efimov, *Tr. Moskovsk. Mat. Obshch.*, 14, 211 (1965).

9. V. I. Ponomarev, *Fundam. Math.*, 52, 351 (1963).

10. R. Engelking, *Outline of General Topology*, Amsterdam, 1968.

11. R. Engelking, *Fundam. Math.*, 57, No. 3, 287 (1965).

Note: Figure translations are in progress. See original paper for figures.