A LINEAR DIFFERENTIAL GAME OF EVASION

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Abstract

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Here a strengthening of the results of paper (1) is given.

A linear differential game is considered

$$\dot{z} = Cz - u + v + a. \tag{1}$$

Here z is the phase vector of the game, belonging to a given vector Euclidean space R of finite dimension; C is a given linear mapping of the space R into itself; a is a given constant vector from R; u is the control of pursuit; v is the control of evasion; u and v are vectors from R, but they are not arbitrary, and satisfy the conditions: $u \in P$, $v \in Q$, where P and Q are given compact convex subsets of the space R. The game is considered finished when z reaches a given vector subspace M of the space R.

The aim of the game is to prevent its termination; for this purpose, at each instant of time t we choose the value $v(t)$ of the control v, using the functions $z(s)$ and $u(s)$, known to us on the interval $0 \leq s \leq t$. Such are the rules of the game.

Let L denote the orthogonal complement in R to M; let ν denote the dimension of L, and let π denote the operation of orthogonal projection from R onto L. Let A and B be two subsets of the space L. We shall write $A \overset{*}{\subset} B$ if there exists a vector $x \in L$ such that $x + A \subset B$.

Evasion theorem. *If $\nu \geq 2$ and there exists a real number $\mu > 1$ such that the relations*

$$\dim \pi e^{\tau C} Q = \nu; \qquad \mu \pi e^{\tau C} P \overset{*}{\subset} \pi e^{\tau C} Q \tag{2}$$

hold for all sufficiently small real positive values of the parameter τ, then, acting according to the rules of the game, we can prevent its termination throughout the whole time $0 \leq t < \infty$, if, of course, the initial state z_0 does not belong to

M. Moreover, we can conduct the game in such a way that for the distance of the point $z(t)$ to M the estimate (3) holds.

To write the estimate, denote by ξ the distance of the point z to M, and by η its distance to L. Then the estimate holds

$$\xi(t) > \frac{c\xi_0^k}{[1+\eta(t)]^m}, \tag{3}$$

provided only that $\xi_0 \leq \varepsilon$. Here c, ε are positive constants, and k and m are natural numbers, depending only on the game, but not on its course.

Let us give a more detailed description of the process of evasion. A parallel translation of either of the sets P and Q in the space R can be compensated by a change of the vector a; using this, we can arrange that the sets P and Q belong respectively to vector subspaces U and V of the space R, with $\dim P = \dim U$, $\dim Q = \dim V$; in addition, for simplicity, let us suppose that $\dim V = \nu$.

Further, we may assume that, instead of (2), the ordinary inclusion holds

$$\mu\pi e^{\tau C}P \subset \pi e^{\tau C}Q. \tag{4}$$

Define the linear mappings f_τ and g_τ, respectively, of the spaces U and V by the formulas

$$f_\tau = \pi e^{\tau C}; \qquad g_\tau = \pi e^{\tau C}. \tag{5}$$

It turns out that the mapping

$$h_\tau = g_\tau^{-1} f_\tau \tag{6}$$

is an analytic function of the parameter τ for all small values of τ, although this is not true for the mapping g_τ^{-1}, so that a linear mapping h_0 of the space U into the space V is defined. It is also possible to achieve, by a parallel shift of the set Q, that there exists a sufficiently small positive number δ such that if a vector $w \in V$ satisfies the inequality $|w| \leq \delta$, then the vector v, defined by (7), belongs to Q:

$$v = h_0(u) + w, \qquad \text{where } u \in P. \tag{7}$$

To each point $z_0 \in R$, for which $\xi_0 \leq 1$, there is assigned a time interval of length

$$\theta = \theta_0/(1+\eta_0), \tag{8}$$

where $\theta_0 > 0$ is a constant depending on the game. In addition, to the same point z_0 there is assigned a value of the vector $w = w(z_0)$, satisfying the condition $|w(z_0)| \leq \delta$, such that the control $v(t)$, defined from the control $u(t)$ on the interval $0 \leq t \leq \theta$ by formula (7), i.e., by the relation

$$v(t) = h_0(u(t)) + w(z_0), \tag{9}$$

together with the control $u(t)$ gives a motion $z(t)$, $0 \leq t \leq \theta$, $z(0) = z_0$, for which the inequalities

$$\xi(\theta) > \varepsilon/[1 + \eta(\theta)]^k, \qquad \varepsilon \leq 1; \tag{10}$$

$$\xi(t) > c' \xi_0^k/[1 + \eta(t)]^{2k-1}, \tag{11}$$

are satisfied, where ε and c' are positive constants depending on the game.

Inequality (10) provides a basis for considering, in the space R, the hypersurface S defined by the equation

$$\xi = \varepsilon/[1 + \eta]^k. \tag{12}$$

The hypersurface S divides the space R into two regions: the inner region S_-, containing M, and the outer region S_+.

If during some part of the game the point $z(t)$ is outside the surface S, then we do not concern ourselves with the choice of the control v, and only at the moment of time t_0, when the point $z(t_0) \in S$, do we switch on, for the time θ (see (8)), the special evasion control $v(t)$ specified by formula (9), taking $z_0 = z(t_0)$. At the end of this time interval the point is again outside the surface S (see (10)), and the process repeats again. On the interval of time $t_0 \leq t \leq t_0 + \theta$ we have the inequality

$$\xi(t) > c\varepsilon^k/[1 + \eta(t)]^{k^2+2k-1}, \tag{13}$$

which is easily derived from inequality (11). In inequality (13) there is a constant c, depending on the game, with $0 < c < c'$. If at the very beginning of the game the point $z_0 = z(0)$ lies inside or on the surface S, then the evasion control (9) is switched on immediately on the time interval $0 \leq t \leq \theta$ (see (8)), and then on this interval inequality (11) holds, while at its end the point $z(\theta)$ is already outside the surface S (see (10)).

Thus, throughout the whole game, with the possible exception of the first time interval $0 \leq t \leq \theta$, the point $z(t)$ either lies outside the surface S, or satisfies inequality (13). On the first time interval it may satisfy inequality (11). Coarsening the inequalities-

(11) and (13), as well as the condition that the point $z(t)$ remains outside the surface S, we obtain estimate (3).

To illustrate the result, let us consider a pursuit process in a Euclidean vector space E of dimension $\nu \geqslant 2$, in which there is a pursuing point x and an evading point y. The motions of these points are given by the equations

$$\ddot{x} + \alpha \dot{x} = u, \qquad |u| \leqslant \rho; \qquad \ddot{y} + \beta \dot{y} = v, \qquad |v| \leqslant \sigma, \tag{14}$$

where $\alpha, \rho, \beta, \sigma$ are positive numbers, and $u, v \in E$ are control vectors; the pursuit process ends when $x = y$. Elementary computations show that if $\sigma > \rho$, then the conditions of the evasion theorem are satisfied, and the point y can at all times move away from the point x. In the case where the opposite inequality $\rho > \sigma$ holds, it follows from the results of note (2) that in the space of initial states $(x_0, \dot{x}_0, y_0, \dot{y}_0)$ there is an open set Ω such that, if the initial state $(x_0, \dot{x}_0, y_0, \dot{y}_0) \in \Omega$, then pursuit always terminates.

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CITED LITERATURE

[1] L. S. Pontryagin, E. F. Mishchenko, DAN, 189, No. 4 (1969).
[2] L. S. Pontryagin, DAN, 174, No. 6 (1967).

Note: Figure translations are in progress. See original paper for figures.