ON THE ERGODIC PRINCIPLE FOR A PARTIALLY SEMIGROUP FAMILY OF OPERATORS ON TOPOLOGICAL SEMIFIELDS

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Abstract

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ON THE ERGODIC PRINCIPLE FOR A PARTIALLY SEMIGROUP FAMILY OF OPERATORS ON TOPOLOGICAL SEMIFIELDS

In this article we shall adhere to the definitions and notation of works ([1],[2]).

1. Let a family $\{E_k,\ 1 \leq k \leq \infty\}$ be given, where E_k is a complete topological semifield; K_k is the cone of nonnegative elements in E_k, and ∇_k is the topological Boolean algebra of all idempotents, on which a certain measure is defined. Elements of E_k for which the integral with respect to this measure exists are called summable; denote their set by L_k. Obviously, L_k is a linear topological space in the topology induced from E_k. An element $x \in L_k \cap K_k$ will be called a distribution if $\mu(x) = 1$, where μ denotes the integral sign.

Introduce the notation

$$P_k = \{x : x \in L_k \cap K_k, \quad \mu(x) = 1\}.$$

Consider, generally speaking, a noncommutative family $\{T_k^{[k+1]},\ 1 \leq k \leq \infty\}$ of linear operators, where $T_k^{[k+1]} : L_k \to L_{k+1}$ and

$$T_k^{[m]} = T_{m-1}^{[m]} \times T_{m-2}^{[m-1]} \cdots T_k^{[k+1]} \quad (k < m).$$

Definition 1. A linear operator $T_k^{[k+1]}$ $(1 \leq k \leq \infty)$ will be called **stochastic** if it satisfies the following conditions:

1) $T_k^{[k+1]}(L_k) \subset L_{k+1}$;

2) $\mu(T_k^{[k+1]}x) = \mu(x)$ for $x \in L_k$;

3) $T_k^{[k+1]}x^{(n)} \to T_k^{[k+1]}x^{(0)}$, if $x^{(n)} \to x^{(0)}$ $(x^{(0)}, x^{(n)} \in L_k)$.

Here convergence is to be understood in the sense of the topology in E_k.

Definition 2. A family $\{T_k^{[k+1]},\ 1 \leq k \leq \infty\}$ of stochastic operators will be called **ergodic** if, for any k and any x, y from P_k and $g \in \nabla_2$, the relation

$$\lim_{r\to\infty} \mu(gT_k^{[r]}(x-y)) = 0$$

holds.

We shall say that a family $\{\widetilde{T}_k^{[k+1]},\ 1 \leq k \leq \infty\}$ of operators includes the family $\{T_k^{[k+1]},\ 1 \leq k \leq \infty\}$ if, for some $i_1, i_2, \ldots, i_h, \ldots$ $(i_h = i_j$ when $h = j)$,

$$\widetilde{T}_{i_h}^{[i_{h+1}]} = T_h^{[h+1]}.$$

Definition 3. A family $\{\widetilde{T}_k^{[k+1]},\ 1 \leq k \leq \infty\}$ of stochastic operators will be called **strongly ergodic** if any family $\{\widetilde{T}_k^{[k+1]},\ 1 \leq k \leq \infty\}$ of stochastic operators that includes the family $\{T_k^{[k+1]},\ 1 \leq k \leq \infty\}$ is ergodic.

Condition A. Whatever x, y from P_k may be, there exist a natural number m, elements $z \in L_m$, u and v from P_m, depending on x, y, and a number $\lambda(T_k^{[m]})$ $(0 < \lambda(T_k^{[m]}) < 1)$ such that

$$T_k^{[m]}x - z = (1 - \lambda(T_k^{[m]}))u,$$

$$T_k^{[m]}y - z = (1 - \lambda(T_k^{[m]}))v,$$

where $z < T_k^{[m]}x,\ z < T_k^{[m]}y,\ k = 1, 2, \ldots$

1.1. If, for the family $\{T_k^{[k+1]},\ 1 \leq k \leq \infty\}$ of stochastic operators, condition (A) is satisfied, then the equality holds

$$T_1^{[n]}(x-y) = (1 - \lambda(T_1^{[m_1]}))(1 - \lambda(T_{m_1}^{[m_2]})) \cdots (1 - \lambda(T_{m_{h-1}}^{[m_h]}))T_{m_h}^{[m_h+r]}(u-v),$$

$$x, y \in P_1, \qquad u, v \in P_{m_h} \qquad \text{for } n = m_1 + m_2 + \cdots + m_h + r, \quad r < m_{h+1} - m_h.$$

1.2. If, for the family $\{T_k^{[k+1]},\ 1 \leq k \leq \infty\}$ of stochastic operators, condition (A) is satisfied and the series

$$\lambda(T_1^{[m_1]}) + \sum_{h=2}^{\infty} \lambda(T_{m_{h-1}}^{[m_h]}) = \infty, \tag{1}$$

then for any x, y from P_1 the relation holds

$$\lim_{n\to\infty} \mu(gT_1^{[n]}(x-y)) = 0.$$

1.3. In order that the family $\{T_k^{[k+1]},\ 1 \leq k \leq \infty\}$ of stochastic operators satisfying condition (A) be strongly ergodic, it is necessary and sufficient that relation (1) hold.

All known criteria of ergodicity for inhomogeneous Markov chains (see ([3]), ([4]), p. 206, ([5],[6])) are sufficient conditions for strong ergodicity and constitute special cases of condition (1). From proposition 1.3, as a consequence, follow the results of papers ([7]) (Theorem 11), ([8]) (Lemma 2), ([9]) (Theorem 1).

2. Consider a topological semifield $E_k(t)$, depending on a numerical parameter t, and denote by $\mathcal{E}_k(t)$ the direct sum

$$\mathcal{E}_k(t) = E_k(t) \oplus iE_k(t).$$

As in item 1, it is assumed that on $\nabla_k(t)$ a certain proper measure is defined.

Let there be given, generally speaking, a noncommutative family of linear operators $\{T_k^{[k+1]}(t),\ 1 \leq k \leq \infty\}$, where $T_k^{[k+1]}(t) : \mathcal{L}_k(t) \to \mathcal{L}_{k+1}(t)$ and

$$T_k^{[m]}(t) = T_{m-1}^{[m]}(t) \times T_{m-2}^{[m-1]}(t) \cdots T_k^{[k+1]}(t)$$

for $k < m$. Here $\mathcal{L}_k(t)$ is the set of summable elements of the complex semifield $\mathcal{E}_k(t)$.

Definition 4. A family $\{T_k^{[k+1]}(t),\ 1 \leq k \leq \infty\}$ of linear operators continuously depending on the numerical parameter t will be called a **family of characteristic operators** if the following relations are fulfilled:

1) $T_k^{[k+1]}(t)(\mathcal{L}_k(t)) \subseteq \mathcal{L}_{k+1}(t)$;
2) $T_k^{[k+1]}(0) = T_k^{[k+1]}$ is a stochastic operator;
3) $T_k^{[k+1]}(t)x^{(n)}(t) \to T_k^{[k+1]}(t)x^{(0)}(t)$, if

$$x^{(n)}(t) \to x^{(0)}(t) \quad (x^{(0)}(t), x^{(n)}(t) \in \mathcal{L}_k(t)).$$

If, for the family $\{T_k^{[k+1]}(t),\ 1 \leq k \leq \infty\}$ of characteristic operators, a condition analogous to (A), taking into account the parameter t, is satisfied, then results 1.1 and 1.2 remain valid for this family.

From the results of item 2, as a consequence, follows the result of paper ([10]) (Lemma 5).

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Note: Figure translations are in progress. See original paper for figures.