BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC AND PARABOLIC EQUATIONS IN UNBOUNDED DOMAINS

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BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC AND PARABOLIC EQUATIONS IN UNBOUNDED DOMAINS

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I. Let G be the exterior of a bounded domain in R^n. The boundary of G is a smooth $(n-1)$-dimensional surface Γ. We shall consider the boundary-value problem

$$A(x,D)u = \sum_{|\alpha|\le 2l} a_\alpha(x) D^\alpha u = f(x), \quad x \in \bar{G}; \tag{1}$$

$$B_j(x,D)u = \sum_{|\beta|\le m_j} b_{j\beta}(x) D^\beta u = g_j(x), \quad x \in \Gamma, \tag{2}$$

where

$$j = 1, \dots, l; \quad \alpha = (\alpha_1, \dots, \alpha_k), \quad |\alpha| = \sum_{k=1}^{n} \alpha_k,$$

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_k^{\alpha_k} = \frac{1}{(-i)^{\alpha_k}} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}.$$

We assume that the following conditions are satisfied:

1. The operator A is uniformly elliptic in $\bar{G}$.
2. At every point $x \in \Gamma$ the Shapiro–Lopatinskii condition is satisfied.
3. The limit $\lim_{|x|\to\infty} a_\alpha(a)$ exists for all α, $|\alpha| \le 2l$.
4. The symbol of the limiting operator

$$a(\infty,\xi) = \sum_{|\alpha|\le 2l} a_\alpha(\infty)\xi^\alpha$$

has no zeros for all $\xi \in R^n$.

5. The coefficients $a_\alpha(x)$ and $b_{j\beta}(x)$ are infinitely smooth functions. The spaces $H^k(G)$ and $H^{k-1/2}(\Gamma)$ are the usual Sobolev-Slobodetskii spaces.

Theorem 1. Suppose that conditions 1-5 are satisfied. Then:

a) for any $u \in H^k(G)$ the inequality

$$\|u\|_k(G) \leq C\left\{\|f\|_{k-2l}(G) + \sum_{j=1}^{l}\|g_j\|_{k-m_j-1/2}(\Gamma) + \|u\|_{k-1}(G_0)\right\},$$

holds, where $k \geq \max(2l, m_j + 1/2)$, $G_0 = G \cap K_R$; $K_R = \{x : |x| \leq R\}$;

b) there exists a regularizer for problem (1), (2), i.e., a bounded operator

$$R : H^{k-2l}(G) \times H^{k-m_j-1/2}(\Gamma) \to H^k(G)$$

such that $LR\Phi = \Phi + T\Phi$, where $L = (A, B_j)$, $\Phi = (f, g_j)$, and T is a completely continuous operator in the direct product

$$H^{k-2l}(G) \times H^{k-m_j-1/2}(\Gamma).$$

The space $H_a^k(G)$ is the space of functions with norm

$$\|u\|_{k,a}(G) = \sum_{|\alpha|\leq k}\iint |D^\alpha u|^2 e^{2a|x|}\,dx.$$

Theorem 2. Suppose that conditions 1-5 are satisfied and that the polynomial $a(\infty, \xi + i\tau)$ has no real zeros in ξ for $|\tau| \leq a$. Then:

a) for any function $u(x) \in H_a^k(G)$ the estimate is valid

$$\|u\|_{k,a}(G) \leq C\left\{\|f\|_{k-2l,a}(G) + \sum_{1}^{l}\|g_j\|_{k-m_j-1/2}(\Gamma) + \|u\|_{k-1}(G_0)\right\};$$

b) there exists a regularizer of the problem (1), (2), acting from the space $H_a^{k-2l}(G) \times H^{k-m_j-1/2}$ into the space $H_a^k(G)$.

II. In the same domain with boundary Γ we consider the elliptic problem with complex parameter

$$A(x, D, q)u(x, q) = \sum_{|\alpha|+\beta\leq 2l} a_{\alpha\beta}(x) q^\beta D^\alpha u = f(x, q), \qquad x \in \bar{G}; \tag{3}$$

$$B_j(x, D, q)u(x, q) = \sum_{|\beta|+|\gamma|\leq m_j} q^\gamma b_{j\beta\gamma}(x) D^\beta u = g_j(x, q), \qquad x' \in \Gamma. \tag{4}$$

The analogous problem in a bounded domain was considered in the work ([1]). The parameter q varies in the angle $Q : \varphi_1 \leq \arg q \leq \varphi_2$. We assume that the following conditions are satisfied:

6. The coefficients $a_{\alpha\beta}(x)$ and $b_{j\beta\gamma}(x)$ are infinitely differentiable functions in $\bar{G}$, and the inequalities $|D^\gamma a_{\alpha\beta}| \le C^\beta_{a,\gamma}$ hold, where $C^\beta_{a,\gamma}$ are constants.
7. The operator A is elliptic with parameter, and the boundary-value problem A, B_j satisfies the Shapiro-Lopatinskii condition with parameter (for the exact definition see Chap. I, § 4 of ([1])).

Theorem 3. *Let conditions 6 and 7 be satisfied and let k be an integer $k \ge k_0 = \max(2l, m_j+1)$. Then there exists a number $q_1 > 0$ such that for $|q| > q_1,\ q \in Q$, problem (3), (4) has a unique solution $u(x) \in H^k(G)$, if $f(x) \in H^{k-2l}(G)$, and $g_j(x) \in H^{k-m_j-1/2}(\Gamma)$. Moreover, the inequality holds*

$$\|u\|_k(G) + |q|^k\|u\|_0(G) \le C\left\{\|f\|_{k-2l} + |q|^{k-2l}\|f\|_0 + \right.$$

$$\left. + \sum_1^l \left(\|g_j\|_{k-m_j-1/2}(\Gamma) + |q|^{k-m_j-1/2}\|g_j\|_0(\Gamma)\right)\right\}.$$

The constant C does not depend on $u(x)$ and q.

Remark. An analogous result is valid for the case of an unbounded domain G with unbounded boundary Γ.

III. In the author' s work ([3]) problems for elliptic equations with unbounded coefficients in the exterior of a bounded domain were considered. Here we shall study the analogous problem with parameter. To this end, assume that problem (3), (4) satisfies the following conditions:

8. The inequality holds

$$\left|\sum_{|\alpha|+\beta=2l} a_{\alpha\beta}(x)\xi^\alpha q^\beta\right| \ge \delta(x)(|\xi|^2 + |q|^2)^l$$

for any $\xi \in R^n,\ q \in Q = \{q : \varphi_1 \le \arg q \le \varphi_2\}$, $\delta(x) \ge \delta_0 > 0$.

9. The function

$$a_{2l}(x,\xi,q) = \sum_{|\alpha|+\beta=2l} a_{\alpha\beta}(x) q^\beta \xi^\alpha$$

for $|x| > N,\ q \in Q,\ \xi \in R^n$ leaves free of values the angle of the complex plane $Q_1 = \{z : \psi_1 \le \arg z \le \psi_2\}$.

10. The values of the function $a_{0,0}(x)$ for $|x| > N$ lie in the angle $Q_2 = \{z : \psi_1 + \pi + \varepsilon \le \arg z \le \psi_2 + \pi - \varepsilon\},\ \varepsilon > 0$.

11. At every point $x' \in \Gamma$ the Shapiro–Lopatinskii conditions with parameter are satisfied.

12. Let x be any point of G, $|x| > N$; H a cube with edge h centered at the point x; y any point of H. We require that the functions $|a_{\alpha\beta}(x)-$

$-a_{\alpha\beta}(y)|/\delta(x)$ for $|\alpha|+\beta = 2l$, $|a_{0,0}(x)-a_{0,0}(y)|/|a_{0,0}(x)|$ tend to zero as $h \to 0$ uniformly in x, $|x| \geq N$, $x \in G$.

Introduce the notation $\delta_k(x) = (\delta(x))^{k/2l}$, $\gamma_k(x) = (|a_{0,0}(x)|)^{k/2l}$.

13. Let $|\alpha|+\beta = 2l-k$, and $|\gamma| = p$; then the ratio $|D^\gamma a_{\alpha\beta}(x)|/\delta_{2l-(k+p)}\gamma_{k+p} \to 0$ as $|x| \to \infty$, if $0 \leq p \leq s-2l$ for $0 < k < 2l$ and $0 < p \leq s-2l$ for $k = 0, 2l$.

Conditions 12 and 13 allow certain classes of unbounded functions. Condition 12 is satisfied by polynomials and functions of the type $e^{k|x|}$.

For convenience we shall use special norms depending on the parameter q. For each fixed q they are equivalent to the norms $\|u\|_k(G)$, which were used by the author. The definition of the norm is as follows:

$$\langle u\rangle_k^2(G) = \iint \{\delta_k^2(x)|D^k u|^2 + |q|^{2k}|u|^2 + \gamma_k^2(x)|u|^2\}\, dx.$$

Theorem 4. *Suppose conditions 8–13 are fulfilled. Then there exists a number* $q_1 > 0$ *such that, for* $|q| > q_1$, $q \in Q$, *there exists a unique solution of problem (3), (4) belonging to the space* $H^k(\delta, \gamma, G)$, *if* $f \in H^{k-2l}(\delta, \gamma, G)$, *and* $g_j \in H^{k-m_j-1/2}(\Gamma)$. *Moreover, the estimate*

$$\langle u\rangle_k(G) \leq C\left\{\langle f\rangle_{k-2l}(G) + \sum_1^l \|g_j\|_{k-m_j-1/2}(\Gamma)\right\}.$$

holds.

The constant C *does not depend on* $u(x)$ *and* q.

An analogous result is also true in the case of an unbounded boundary.

IV. After studying the elliptic problem with a parameter, it is natural to investigate the general mixed problem for a parabolic equation in a cylinder with unbounded base. Let G, as before, be the exterior of a bounded domain with boundary Γ, and let Ω_T be the cylinder $\Omega_T = G \times (0, T)$ with lateral surface $\Omega'_T = \Gamma \times (0, T)$.

Consider the parabolic problem

$$A(x, t, D_x, \partial/\partial t)u(x, t) = f(x, t), \qquad (x, t) \in \Omega_T; \tag{5}$$

$$B_j(x,t,D_x,\partial/\partial t)u(x,t)\big|_{x=x'} = g_j(x',t), \qquad (x',t)\in\Omega'_T;\ j=1,\dots,l. \tag{6}$$

For $t=0$ we prescribe the initial conditions

$$\partial^k u/\partial t^k\big|_{t=0} = \varphi_k(x) \qquad (k=0,\dots,\chi-1,\ x\in G),\ \chi = l/b. \tag{7}$$

The operator A has order $2l$, and the operators B_j have order m_j. The parabolic weight of the problem is $2b$.

Definition. Problem (5), (6) is called **normally parabolic** if, for each fixed $t=a$, $0\le a\le T$, the problem with a parameter in the domain G, obtained from problem (5), (6) by replacing $\partial/\partial t$ with q^{2b}, satisfies all conditions 8–13 in the angle $|\arg q|\le \pi/4b$.

We shall use on the surface Ω'_T the usual Sobolev–Slobodetskii space $H_{\lambda,\lambda/2b}(\Omega'_T)$, and in the cylinder Ω_T the space $H_{s,s/2b}(\delta,\gamma,\Omega_T)$ with norm

$$\langle u\rangle^2_{s,s/2b} = \int_{\Omega_T}\!\!\int \{\delta_s^2(x)(|D_x^s u|^2 + |D_t^{s/2b}u|^2) + \gamma_s^2(x)|u|^2\}\,dx\,dt.$$

It is assumed that s, $s/2b$ are integers.

The discussion is carried out by the method of the Laplace transform, analogously to how this was done in paper ([1]).

Theorem 5. *Let problem (5), (6), (7) be a normally parabolic problem. Suppose also that the usual compatibility conditions for the right-hand sides are fulfilled. Then, if $f(t,t)\in H_{\mu,\mu/2b}(\delta,\gamma,\Omega_T)$; $g_j(x',t)\in H_{\lambda_j,\lambda_j/2b}(\Omega'_T)$;*

$\varphi_k(x)\in H_{s-2bk-b}(G)$, $k=0,\dots,\varkappa-1$, where $\mu = s-2l$, $\lambda_j = s-m_j-\frac{1}{2}$, then there exists a unique solution of the problem $u(x,t)\in H_{s,s/2b}(\delta,\gamma,\Omega_T)$. Moreover, the a priori estimate holds

$$\langle u\rangle_{s,s/2b} \le C\left\{\langle f\rangle_{\mu,\mu/2b} + \sum_1^l \|g_j\|_{\lambda_j,\lambda_j/2b} + \sum_1^{\varkappa-1}\|\varphi_k\|_{s-2bk-b}\right\}.$$

The constant C does not depend on $u(x,t)$.

V. Now in the same domain G with boundary Γ consider problem (1), (2). We shall assume that conditions 1, 2, and 5 are satisfied. Conditions 3 and 4 will be replaced by the following.

Condition 9°. The coefficients of equation (1) and their derivatives have the form

$$D^\beta a_\alpha(x) = a^0_{\alpha\beta}(\omega) r^{|\alpha|-|\beta|-2l} + a^1_{\alpha\beta}(x); \quad r=|x|.$$

Here $\omega = (\omega_1, \dots, \omega_{n-1})$ is an infinitely differentiable coordinate system on the unit sphere, and

$$a^1_{\alpha\beta}(x) = O(r^{|\alpha|-|\beta|-2l-\delta})$$

as $r \to \infty$; $\delta > 0$.

Consider in R^n the elliptic equation $\sum a^0_\alpha(\omega) D^\alpha u = f$. Make the change of variables $x \to (r, \omega)$, and then the change $r = e^{-t}$. It is well known that after these changes we obtain the equation

$$A^0(\omega, D_\omega, D_t) v(t, \omega) = e^{-2lt} f(t, \omega) \tag{8}$$

in the cylinder $\Omega = S \times (-\infty, +\infty)$, where S is the unit sphere $|x| = 1$. After the Fourier transform in t we obtain an elliptic equation with a parameter on the sphere:

$$A^0(\omega, D_\omega, \lambda)\tilde{v}(\lambda, \omega) = \tilde{f}_1(\lambda, \omega). \tag{9}$$

It is known that the inverse operator $R(\lambda)$ for equation (9) is a meromorphic function of λ, and in each strip $|\operatorname{Im}\lambda| < C_1$ there exists only a finite number of poles. The spaces $\dot{W}^k_\alpha(G)$ are the same as those used by V. A. Kondrat' ev in paper (2).

Theorem 6. Suppose conditions 1, 2, 5, and 9 are satisfied, and suppose that the function $R(\lambda)$ of problem (9) has no poles on the line

$$\operatorname{Im}\lambda = \frac{1}{2}(-\alpha - n + 2k), \qquad k \geq k_0 = \max(2l, m_j + \tfrac{1}{2}).$$

Then, if

$$(f, g_j) \in \dot{H}^k_\alpha = \dot{W}^{k-2l}_\alpha \times H^{k-m_j-1/2},$$

then, provided a finite number of conditions of the form $F_i(f, g_j) = 0$ are satisfied, where F_i are functionals in $\dot{H}^k_\alpha(G)$, there exists a solution of problem (1), (2) belonging to $\dot{W}^k_\alpha(G)$.

Under the same assumptions, there hold a theorem on the increase of smoothness, a theorem on the finite-dimensionality of the space of solutions of the homogeneous problem, and a theorem on the asymptotic expansion of the solution as $r \to \infty$. The methods of proof are analogous to the methods of proof in paper (2).

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Note: Figure translations are in progress. See original paper for figures.