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**Abstract**

**Full Text**

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*MATHEMATICS*

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## THE CONJUGATION PROBLEM FOR THREE ANALYTIC FUNCTIONS

*(Presented by Academician N. I. Muskhelishvili on May 4, 1970)*

Let  $L$  be a simple smooth closed contour dividing the complex plane into an interior domain  $S^+$ , containing the point  $z = 0$ , and an exterior domain  $S^-$ , containing the point at infinity. Let us construct one more exterior domain  $E^-$  as follows. Join the points  $z = 0$  and  $z = \infty$  by a simple cut intersecting  $L$  at the single point  $t_0$ . In what follows, by  $\sqrt{z}$  we shall mean the fixed branch having a discontinuity along the indicated cut. The locus of the points  $\sqrt{t}$  and  $(-\sqrt{t})$ , where  $t$  runs over  $L$ , forms a simple smooth closed contour  $\Gamma$ . As  $E^-$  we take the domain lying outside this contour. It is obvious that if  $z \in E^-$ , then also  $(-z) \in E^-$ .

**Problem.** Find functions  $\Phi^+(z)$  and  $\Psi^+(z)$ , analytic in  $S^+$  and continuous in  $S^+ + L$ , and also a function  $\Phi^-(z)$ , analytic in  $E^-$ , continuous in  $E^- + \Gamma$ , and equal to zero at infinity, under the boundary conditions

$$\begin{aligned} \Phi^+(t) &= A(t)\Phi^-(\sqrt{t}) + G(t), \\ \Psi^+(t) &= B(t)\Phi^-(\sqrt{t}) + H(t), \end{aligned} \quad t \in L, \quad (1)$$

where  $A(t), B(t), G(t)$ , and  $H(t)$  are functions given on  $L$  satisfying a Hölder condition;  $A(t)$  and  $B(t)$  do not vanish anywhere on  $L$ .

Below it is shown that problem (1), in its nature and in the method of solution, is related to the well-known conjugation problem (the Riemann problem) <sup>(1,2)</sup>.

1. Let us first consider the "jump" problem

$$\begin{aligned} \varphi^+(t) &= \varphi^-(\sqrt{t}) + g(t), & \psi^+(t) &= \varphi^-(\sqrt{t}) + h(t), \\ t \in L, & & \varphi^-(\infty) &= 0. \end{aligned} \quad (2)$$

Introduce the new unknown functions

$$\varphi_1^-(z) = \varphi^-(\sqrt{z}) - \varphi^-(-\sqrt{z}), \quad \varphi_2^-(z) = 1/\sqrt{z} [\varphi^-(\sqrt{z}) - \psi^-(-\sqrt{z})],$$

$$z \in S^-.$$

These functions are analytic in  $S^-$ , continuous in  $S^- + L$ , and equal to zero at infinity. Setting further  $\varphi_1^+ = \varphi^+ + \psi^+$ ,  $\varphi_2^+ = \varphi^+ - \psi^+$ , from conditions (2) we arrive at two independent conjugation problems

$$\varphi_1^+(t) = \varphi_1^-(t) + g(t) + h(t), \quad t \in L; \quad (3)$$

$$\varphi_2^+(t) = \sqrt{t} \varphi_2^-(t) + g(t) - h(t), \quad t \in L. \quad (4)$$

Solving problems (3) and (4) by the known method and using the results of N. I. Muskhelishvili on the behavior of a Cauchy-type integral with density having a discontinuity (see, in particular, <sup>(1)</sup>, p. 77), we arrive at the theorem.

**Theorem 1.** Let the given function  $g(t) + h(t)$  satisfy the Hölder condition everywhere on  $L$ , and let the given function  $g(t) - h(t)$  satisfy

satisfies this condition on  $L$  everywhere except, possibly, the point  $t_0$ , where a discontinuity of the first kind is allowed for this function.

Then problem (2) has, in the class of functions analytic in the above-indicated domains and continuous up to the boundary, a unique solution ( $L$  is traversed counterclockwise):

$$\varphi^+(z) = \frac{1}{4\pi i} \int_L \frac{g(\tau) + h(\tau)}{\tau - z} d\tau + \frac{\sqrt{z - t_0}}{4\pi i} \int_L \frac{g(\tau) - h(\tau)}{\sqrt{\tau - t_0}(\tau - z)} d\tau, \quad z \in S^+; \quad (5)$$

$$\psi^+(z) = -\frac{1}{4\pi i} \int_L \frac{g(\tau) + h(\tau)}{\tau - z} d\tau - \frac{\sqrt{z - t_0}}{4\pi i} \int_L \frac{g(\tau) - h(\tau)}{\sqrt{\tau - t_0}(\tau - z)} d\tau, \quad z \in S^+; \quad (6)$$

$$\varphi^-(z) = \frac{1}{4\pi i} \int_L \frac{g(\tau) + h(\tau)}{\tau - z^2} d\tau + \frac{z\sqrt{z^2 - t_0}}{4\pi i\sqrt{z^2}} \int_L \frac{g(\tau) - h(\tau)}{\sqrt{\tau - t_0}(\tau - z^2)} d\tau, \quad z \in E^-. \quad (7)$$

The limiting values  $\varphi^+(t)$ ,  $\psi^+(t)$ , and  $\varphi^-(t)$  satisfy the Hölder condition.  $\sqrt{w - t_0}$  denotes the branch of the root having a discontinuity along a part of the cut indicated earlier, connecting  $t_0$  with infinity.

2. We proceed to the problem of factorization of the coefficients  $A(t)$  and  $B(t)$ . By analogy with (5)–(7), we write formulas in which the logarithms are single-valued functions, having, possibly, discontinuities at the single point  $t_0$ :

$$X_0^+(z) = \exp \left\{ \frac{1}{4\pi i} \int_L \frac{\ln A(\tau)B(\tau)}{\tau - z} d\tau + \frac{\sqrt{z - t_0}}{4\pi i} \int_L \ln \frac{A(\tau)}{B(\tau)} d\tau / \sqrt{\tau - t_0} \times (\tau - z) \right\}, \quad z \in S^+; \quad (8)$$

$$\Omega_0^+(z) = \exp \left\{ \frac{1}{4\pi i} \int_L \frac{\ln A(\tau)B(\tau)}{\tau - z} d\tau - \frac{\sqrt{z - t_0}}{4\pi i} \int_L \ln \frac{A(\tau)}{B(\tau)} d\tau / \sqrt{\tau - t_0} \times (\tau - z) \right\}, \quad z \in S^+; \quad (9)$$

$$X_0^-(z) = \exp \left\{ \frac{1}{4\pi i} \int_L \frac{\ln A(\tau)B(\tau)}{\tau - z} d\tau + \frac{z\sqrt{z^2 - t_0}}{4\pi i\sqrt{z^2}} \int_L \ln \frac{A(\tau)}{B(\tau)} d\tau / \sqrt{\tau - t_0} \times (\tau - z^2) \right\}, \quad z \in E^-. \quad (10)$$

The properties of the functions  $X_0^+$ ,  $\Omega_0^+$ , and  $X_0^-$  depend on the index

$$\varkappa = \text{Ind } A(t) + \text{Ind } B(t) = \frac{1}{2\pi i} [\ln A(\tau)B(\tau)]_L. \quad (11)$$

Let the function  $(z - t_0)^{\varkappa/2}$  be analytic in  $S^+$ , and let the function  $\omega(z) = (z^2 - t_0)^{\varkappa/2}$  be analytic in  $E^-$ , with the possible exception of the point  $z = \infty$ , and satisfy the condition  $\omega(1/\bar{t}) = (t - t_0)^{\varkappa/2}$  for  $t \in L$ . On the basis of formulas of N. I. Muskhelishvili ((1), § 22) we obtain

**Theorem 2.** The functions

$$X^+(z) = (z - t_0)^{\varkappa/2} X_0^+(z), \quad \Omega^+(z) = (-1)^{\varkappa} (z - t_0)^{-\varkappa/2} \Omega_0^+(z) \quad (12)$$

are analytic in  $S^+$  and continuous in  $S^+ + L$ ; the limiting values  $X^+(t)$  and  $\Omega^+(t)$  satisfy the Hölder condition on  $L$ .

The function

$$X^-(z) = (z^2 - t_0)^{-\varkappa/2} X_0^-(z) \quad (13)$$

is analytic in  $E^-$  (for  $\varkappa < 0$  it has a pole of order  $|\varkappa|$  at infinity), and is continuously extendable to the contour  $\Gamma$ ; the limiting value  $X^-(t)$  satisfies the Hölder condition on  $\Gamma$ .

The formulas are valid

$$X^+(t) = A(t)X^-(\sqrt{t}), \quad \Omega^+(t) = B(t)X^-(-\sqrt{t}), \quad t \in L. \quad (14)$$

The functions (12) and (13) play the role of the canonical solution of the conjugation problem. These functions do not vanish anywhere at a finite distance.

**3.** Let us proceed to the direct solution of problem (1). Using the factorization (14), we represent condition (1) in the form

$$\Phi^+(t)/X^+(t) = \Phi^-(\sqrt{t})/X^-(\sqrt{t}) + G(t)/X^+(t),$$

$$\Psi^+(t)/\Omega^+(t) = \Phi^-(-\sqrt{t})/X^-(-\sqrt{t}) + H(t)/\Omega^+(t), \quad t \in L. \quad (15)$$

Consider the unknown function  $\Phi^-(z)/X^-(z)$ . On the basis of the properties of the function  $X^-(z)$  and of the condition  $\Phi^-(\infty) = 0$  adopted by us, we conclude that the function  $\Phi^-(z)/X^-(z)$  for  $\varkappa \leq 0$  has at infinity a zero of order not less than  $|\varkappa| + 1$ , while for  $\varkappa > 0$  this function may have a pole of order not greater than  $\varkappa - 1$ . Therefore the indicated function can be represented in the form

$$\Phi^-(z)/X^-(z) = \varphi^-(z) + P_{\varkappa-1}(z), \quad (16)$$

where  $\varphi^-(z)$  vanishes at infinity, and  $P_{\varkappa-1}(z)$  is a polynomial of degree not exceeding  $\varkappa - 1$ . For  $\varkappa \leq 0$  we put  $P_{\varkappa-1} \equiv 0$ .

Introducing also the notation

$$\Phi^+(z)/X^+(z) = \varphi^+(z), \quad \Psi^+(z)/\Omega^+(z) = \psi^+(z), \quad (17)$$

we give conditions (15) the form of the jump problem (2), where now

$$g(t) = G(t)/X^+(t) + P_{\varkappa-1}(\sqrt{t}), \quad h(t) = H(t)/\Omega^+(t) + P_{\varkappa-1}(-\sqrt{t}). \quad (18)$$

Using Theorem 1, we arrive at the following results.

**Theorem 3.** *If the index  $\varkappa$  defined by formula (11) is positive, then problem (1) is unconditionally solvable, and the homogeneous problem has exactly  $\varkappa$  linearly independent solutions. The general solution can be constructed in quadratures by formulas (16), (17), (5)–(7), (18), (12), (13), and (8)–(10), where  $P_{\varkappa-1}(z)$  is a polynomial of degree  $\varkappa - 1$  with arbitrary complex coefficients.*

**Theorem 4.** *In the case  $\varkappa = 0$ , problem (1) is unconditionally solvable and has a unique solution. This solution can be constructed in quadratures by the formulas indicated in Theorem 3, where  $P_{\varkappa-1} \equiv 0$ .*

As an example, let us write out the function  $\Phi^-(z)$ . Using equality (16), where  $P_{\varkappa-1} \equiv 0$ , we obtain

$$\Phi^-(z) = X^-(z) \left\{ \frac{1}{4\pi i} \int_L \left( \frac{G(\tau)}{X^+(\tau)} + \frac{H(\tau)}{\Omega^+(\tau)} \right) \frac{d\tau}{\tau - z^2} + \frac{z\sqrt{\bar{z}^2 - t_0}}{4\pi i\sqrt{\bar{z}^2}} \int_L \left( \frac{G(\tau)}{X^+(\tau)} - \frac{H(\tau)}{\Omega^+(\tau)} \right) \frac{d\tau}{\sqrt{\tau - t_0}(\tau - z^2)} \right\}, \quad z \in E^-. \quad (19)$$

We shall also arrive at an expression of the form (19) when solving problem (1) in the case  $\varkappa < 0$ . However, in the latter case the function  $X^-(z)$  has at infinity a pole of order  $|\varkappa|$ , as a result of which problem (1) will be solvable not for all functions  $G(t)$  and  $H(t)$  satisfying the Hölder condition. In the usual way (cf. (2), pp. 118–119) we derive the necessary and sufficient solvability conditions:

$$\int_L (G(\tau)/X^+(\tau) + H(\tau)/\Omega^+(\tau)) \tau^{k-1} d\tau = 0, \quad k = 1, 2, \dots, -[(\varkappa + 1)/2]. \quad (20)$$

$$\int_L \left( \frac{G(\tau)}{X^+(\tau)} - \frac{H(\tau)}{\Omega^+(\tau)} \right) \left\{ \frac{d^{k-1}}{dz^{k-1}} \frac{\sqrt{1 - t_0 z}}{1 - \tau z} \right\}_{z=0} \frac{d\tau}{\sqrt{\tau - t_0}} = 0,$$

$$k = 1, 2, \dots, [(-\varkappa + 1)/2]. \quad (21)$$

Here the symbol  $[x]$  denotes the greatest integer not exceeding  $x$ .

**Theorem 5.** *In the case  $\varkappa < 0$ , for the solvability of problem (1) the conditions (20) and (21), whose number is equal to  $|\varkappa|$ , are necessary and sufficient. If these conditions are fulfilled, then the problem has a unique solution, which can be obtained from the formulas indicated in Theorem 3, where  $P_{\varkappa-1} \equiv 0$ .*

Let us note that the limiting values  $\Phi^+(t)$ ,  $\Psi^+(t)$ , and  $\Phi^-(t)$  of the functions obtained as a result of solving problem (1) satisfy the Hölder condition and, moreover,  $\Phi^-(\overline{\gamma t_0}) = \Phi^-(\overline{-\gamma t_0})$ .

4. The problem considered is a particular case of the following problem of conjugation of analytic functions, whose number is equal to  $n + 1$ . Suppose that on different copies of the complex plane there are given  $(n + 1)$  domains  $S_1^+, \dots, S_n^+$  and  $E^-$ , bounded by simple smooth closed contours  $L_1, \dots, L_n$  and  $\Gamma$ . Any of the indicated domains may contain the infinitely distant point. On the contours a positive direction of traversal is specified so that the corresponding domains remain on the left. We divide  $\Gamma$  into

the sum of  $n$  simple open arcs  $\gamma_k$ ,  $k = 1, \dots, n$ , and on the contours  $L_k$  we specify one point  $t_k$ ,  $k = 1, \dots, n$ . Let  $\alpha_k(t)$  be functions defined on  $L_k$ , one-to-one (with the exception of  $t_k$ ) and continuously mapping  $L_k$  onto  $\gamma_k$  with a change of the direction of traversal,  $k = 1, \dots, n$ . The points  $t_k$  correspond to the endpoints of the arcs  $\gamma_k$ . It is required to find functions  $\Phi_1^+(z), \dots, \Phi_n^+(z)$ , analytic in the domains  $S_1^+, \dots, S_n^+$ , respectively, continuous in  $S_k^+ + L_k$ , and also a function  $\Phi^-(z)$ , analytic in  $E^-$ , continuous in  $E^- + \Gamma$ , from  $n$  boundary conditions

$$\Phi_k^+(t) = A_k(t)\Phi^-[ \alpha_k(t) ] + G_k(t), \quad t \in L_k, \quad k = 1, \dots, n. \quad (22)$$

Here  $A_k(t)$  and  $G_k(t)$  are given functions on  $L_k$ , satisfying the Hölder condition,  $k = 1, \dots, n$ . It is also assumed that there exist derivatives  $\alpha'_k(t)$  satisfying the Hölder condition and having no zeros.

In the case when all the domains  $S_k^+$  coincide, and the functions  $\alpha_k(t)$  are formed from different branches of the function  $z^{1/n}$ , problem (22), like problem (1), admits a solution in quadratures.

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## CITED LITERATURE

1. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1968.
2. F. D. Gakhov, *Boundary Value Problems*, Moscow, 1963.

*Note: Figure translations are in progress. See original paper for figures.*

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